

MÖBIUS TRANSFORMATIONS AND THE CONFIGURATION SPACE OF A HILBERT SNAKE

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ABSTRACT. The purpose of this paper is to give a simpler proof to the problem of controllability of a Hilbert snake [13]. Using the action of the Möbius group of the unite sphere on the configuration space, in the context of a separable Hilbert space. We give a generalization of the Theorem of accessibility contained in [9] and [14] for articulated arms and snakes in a finite dimensional Hilbert space.

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1. INTRODUCTION AND RESULTS

The group of Möbius transformations of a finite dimensional space is generated by inversions of spheres. It is one of the fundamental geometrical groups. Möbius transformations preserve spherical shapes and also the angles between pairs of curves. This group can be considered as the conformal group of the sphere identified with the compactification of a finite dimensional space.

If we denote by $\mathfrak{M}(\mathbb{S}^n)$ the Möbius transformations of such a sphere \mathbb{S}^n which preserve the orientation, it is known that $\mathfrak{M}(\mathbb{S}^n)$ is isomorphic to the group $SO_0(n, 1)$ which is the connected component of the identity of $O(n, 1)$. All these results can be generalized to the context of a Hilbert space (cf. [3] and [11] for instance). Therefore the group $\mathfrak{M}(\mathbb{S}_{\mathbb{H}})$ of Möbius transformations of the unit sphere $\mathbb{S}_{\mathbb{H}}$ of a Hilbert space \mathbb{H} is also isomorphic to some subgroup $SO_0(\mathbb{H}, 1)$ of the group $O(\mathbb{H}, 1)$ of linear Lorentz transformations of a Lorentz structure on $\mathcal{H} = \mathbb{R} \oplus \mathbb{H}$ (for more details see Subsection 2.2). If we consider $SO(\mathbb{H}, 1)$ as a subgroup of the group $GL(\mathcal{H})$ of continuous automorphisms of \mathcal{H} , we can look for the intersection $SO_{HS}(\mathbb{H}, 1)$ of $SO(\mathbb{H}, 1)$ with the subgroup $GL_{HS}(\mathcal{H})$ of Hilbert-Schmidt automorphisms of $GL(\mathcal{H})$. According to [7], $SO_{HS}(\mathbb{H}, 1)$ can be seen as a limit of an increasing sequence

$$SO(\mathbb{H}_2, 1) \subset \cdots \subset SO(\mathbb{H}_n, 1) \subset \cdots \subset SO_{HS}(\mathbb{H}, 1) \subset GL_{HS}(\mathcal{H}).$$

Via the previous isomorphism from $SO(\mathbb{H}, 1)$ to $\mathfrak{M}(\mathbb{S}_{\mathbb{H}})$ we obtain a subgroup $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ of the group of Möbius transformations of the unit sphere $\mathbb{S}_{\mathbb{H}}$.

On the other hand, as in the finite dimensional situation, the Lie algebra \mathfrak{g} of $SO_{HS}(\mathbb{H}, 1)$ has a decomposition of type $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ where \mathfrak{s} is the Lie algebra of the subgroup $SO_{HS}(\mathcal{H})$ of the Hilbert-Schmidt isometries of the Hilbert space \mathcal{H} . Again \mathfrak{h} can be obtain as an adequate limit of finite dimensional subspace \mathfrak{h}_n which is a factor of the classical decomposition $\mathfrak{g}_n = \mathfrak{h}_n \oplus \mathfrak{s}_n$ of the Lie algebra \mathfrak{g}_n of $SO(\mathbb{H}_n, 1)$. In this way we get a natural sub-Riemannian structure on $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ at the same time directly from \mathfrak{h} and as limit of the canonical sub-Riemannian structure on each $SO(\mathbb{H}_n, 1)$. Now, we know that in the finite dimensional case, each pair of elements of $SO(\mathbb{H}_n, 1)$ can be joined by a horizontal path. Unfortunately, this is no longer true in $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$. Our first result is to proved that there exists in $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ a dense subgroup $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$, provided with its own Lie Banach group structure, such that each pair of elements of $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$ can be joined by a horizontal path (cf. Theorem 3.3.1). This Theorem allows us to give a simpler proof of the accessibility result in the problem of a Hilbert snake obtained in [13].

More precisely, recall that a Hilbert snake of length L is a continuous piecewise C^1 -curve $S : [0, L] \rightarrow \mathbb{H}$, arc-length parameterized such that $S(0) = 0$. Given a fixed partition \mathcal{P} of $[0, L]$, the set $\mathcal{C}_{\mathcal{P}}^L$ of such curves will be called the configuration set and carries a natural structure of Banach manifold. To any "configuration" $u \in \mathcal{C}_{\mathcal{P}}^L$ is naturally associated the end map: $\mathcal{E}(u) = \int_0^L u(s)ds$. This map is smooth and its kernel has a canonical complemented subspace which gives rise to a closed distribution \mathcal{D} on $\mathcal{C}_{\mathcal{P}}^L$. The problem of controllability of the "head" $S(L)$ of a Hilbert snake can be transformed in the following accessibility problem in $\mathcal{C}_{\mathcal{P}}^L$ (cf. section 4.4):

Given an initial (resp. final) configuration u_0 (resp. u_1) in $\mathcal{C}_{\mathcal{P}}^L$, such that $\mathcal{E}(u_i) = x_i$, $i = 0, 1$, find a piecewise C^1 horizontal curve $\gamma : [0, T] \rightarrow \mathcal{C}_{\mathcal{P}}^L$ (i.e. γ is tangent to \mathcal{D}) and which joins u_0 to u_1 .

Therefore, given any configuration $u \in \mathcal{C}_{\mathcal{P}}^L$ we have to look for the accessibility set $\mathcal{A}(u)$ of all configurations $v \in \mathcal{C}_{\mathcal{P}}^L$ which can be joined from u by a piecewise C^1 horizontal curve. It is shown in [13] that there exists a canonical

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distribution $\bar{\mathcal{D}}$ which contains the previous horizontal distribution \mathcal{D} which is integrable and each accessibility set $\mathcal{A}(u)$ is a dense subset of the maximal integral manifold of $\bar{\mathcal{D}}$ which contains u .

As in the finite dimensional case (see [9] and [14]), we have a natural action \mathfrak{A} of the group $\mathfrak{M}(\mathbb{S}_{\mathbb{H}})$ on $\mathcal{C}_{\mathcal{P}}^L$. Since we have a canonical isomorphism between $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ and $SO_{HS}(\mathbb{H}, 1)$, let $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$ be the subgroup of $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ which is associated to $SO_{HS}^1(\mathbb{H}, 1) \subset SO_{HS}(\mathbb{H}, 1)$. Then we have the following result

Theorem 1.

- (1) *The orbit through $u \in \mathcal{C}_{\mathcal{P}}^L$ of the restriction of the action \mathfrak{A} to $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ is exactly the maximal integral manifold $\mathcal{L}(u)$ of $\bar{\mathcal{D}}$ which contains u .*
- (2) *The orbit $\mathcal{A}^1(u)$ through $u \in \mathcal{C}_{\mathcal{P}}^L$ of the restriction of the action \mathfrak{A} to $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$ is contained in $\mathcal{A}(u)$ and it is a dense subset of $\mathcal{L}(u)$. In particular $\mathcal{A}(u)$ is a dense subset of $\mathcal{L}(u)$.*

This paper is organized as follows. Section 2 contains in its first part all the definitions and results about Möbius transformations in the Hilbert space context which are needed to prove the announced results of the sub-Riemannian structure on $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$. Properties of the Hilbert-Schmidt group $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ are described in Section 3. In Section 4, according to [13] we first recall all the context concerning the problem of controllability of a Hilbert snake. Then we apply the results of Section 2 to prove Theorem 1. Finally some technical proofs used in Section 2 are presented in Section 5.

2. MÖBIUS TRANSFORMATIONS OF A HILBERT SPACE

2.1. Möbius transformations.

In this paper \mathbb{H} is a fixed Hilbert space on \mathbb{R} and $\{e_i\}_{i \in I}$ will denote a Hilbert basis of \mathbb{H} where I is either the finite set $\{1, \dots, n\}$ with $n \geq 2$ or $I = \mathbb{N} \setminus \{0\}$ and we denote by $\langle \cdot, \cdot \rangle$ the inner product and $\|\cdot\|$ the associated norm. With these notations, we can identify \mathbb{H} with $l^2(I)$ and each $x \in \mathbb{H}$ is identified with the sequence (x_i) where $x_i = \langle x, e_i \rangle$, $i \in I$.

Let \mathbf{H} be any hyperplane in \mathbb{H} . We can always choose a Hilbert basis $\{e_i\}_{i \in I}$ of \mathbb{H} such that $\{e_i\}_{i > 1}$ is a Hilbert basis of \mathbf{H} and we also denote by $\langle \cdot, \cdot \rangle$, $\|\cdot\|$ the induced inner product in \mathbf{H} and $\|\cdot\|$ the associated norm. With these notations, we consider the set $\hat{\mathbf{H}} = \mathbf{H} \cup \{\infty\}$ equipped with the following topology: $U \subset \hat{\mathbf{H}}$ is an open set if and only if $U \cap \mathbf{H}$ is an open set and $\mathbf{H} \setminus U$ is a bounded set in \mathbf{H} , if $\infty \in U$. In this section we will recall the classical properties of the Möbius transformations of \mathbf{H} which the reader can find in ([3], [2] and [11]). We first introduce the following notations:

- Given $a \in \mathbf{H}$ and $r, t \in \mathbb{R}$ with $r > 0$, a Möbius sphere in $\hat{\mathbf{H}}$ is either a classical sphere in \mathbf{H} :

$$(2.1.1) \quad S(a, r) = \{x \in \mathbf{H} : |x - a| = r\},$$

or an extended hyperplane :

$$(2.1.2) \quad P(a, t) = \{x \in \mathbf{H} : \langle x, a \rangle = t\} \cup \{\infty\}.$$

- A reflection in a Möbius sphere S is a transformation in $\hat{\mathbf{H}}$ which is either:

$$\rho(x) = a + \frac{r^2(x - a)}{|x - a|^2}, \quad \rho(a) = \infty \text{ and } \rho(\infty) = a,$$

if S is of type $S(a, r)$, or:

$$\rho(x) = x + \frac{2(t - \langle a, x \rangle)}{|a|^2}a, \quad \rho(\infty) = \infty,$$

if S is of type $P(a, t)$.

- An orthogonal transformation of \mathbf{H} is a linear map $\omega : \mathbf{H} \rightarrow \mathbf{H}$ such that

$$|\omega(x) - \omega(y)| = |x - y| \quad \text{for all } x, y \in \mathbf{H}.$$

- A similitude in $\hat{\mathbf{H}}$ is a transformation σ such that

$$\sigma(x) = \alpha\omega(x) + a, \quad \rho(\infty) = \infty$$

where ω is an orthogonal transformation of \mathbf{H} , $\alpha \in \mathbb{R}$ and a fixed $a \in \mathbf{H}$.

Definition 2.1.1. [11] *A Möbius transformation of $\hat{\mathbf{H}}$ is a bijection on $\hat{\mathbf{H}}$, which is a finite composition of reflections and similitudes.*

We have then the following characterizations:

Theorem 2.1.1. [11]

- (1) A bijection ϕ of $\widehat{\mathbb{H}}$ is a Möbius transformation of $\widehat{\mathbb{H}}$ if and only if the image and the inverse image by ϕ of Möbius spheres are Möbius spheres.
- (2) A Möbius transformation ϕ is a similitude if and only if $\phi(\infty) = \infty$.

Among the set of reflections, we have a particular one which is the reflection in $S(0, 1)$ i.e.

$$\rho_0(x) = \frac{x}{|x|^2} \text{ and } \rho_0(0) = \infty, \rho_0(\infty) = 0.$$

We have then:

Proposition 2.1.1. [3] Let μ be a Möbius transformation. If $\mu(S(0, 1)) = S(a, r)$ then $\mu \circ \rho_0 \circ \mu^{-1}$ is a reflection in $S(a, r)$. If $\mu(S(0, 1)) = P(a, t) \cup \{\infty\}$ then, $\mu \circ \rho_0 \circ \mu^{-1}$ is the reflection in $P(a, t)$.

Given a Möbius sphere S , if $S = S(a, r)$ the two sets

$$S^-(a, r) = \{x \in \mathbb{H} : |x - a|^2 < r^2\}$$

$$S^+(a, r) = \{x \in \mathbb{H} : |x - a|^2 > r^2\} \cup \{\infty\}$$

are called the **two sides** of S . In the same way, if $S = P(a, t) = \{x \in \mathbb{H} : \langle x, a \rangle > t\} \cup \{\infty\}$, the sets :

$$P^-(t, a) = \{x \in \mathbb{H} : \langle a, x \rangle < t\}$$

$$P^+(t, a) = \{x \in \mathbb{H} : \langle a, x \rangle > t\}$$

are the two sides of $P(a, t)$.

Proposition 2.1.2. [3] Let S_1 and S_2 be the two sides of a Möbius sphere S . If μ is a Möbius transformation, then $\mu(S_1)$ and $\mu(S_2)$ are the two sides of the Möbius sphere $\mu(S)$. Moreover, if Σ is one side of S then $\mu(\Sigma) = \Sigma$ implies $\mu(S) = S$.

Let $\{x_1, \dots, x_n\}$ be a family of linearly independent vectors in \mathbb{H} and $x \in \mathbb{H}$. An n -hyperplane P_n in \mathbb{H} is a set of type:

$$\{x + \lambda_1 x_1 + \dots + \lambda_n x_n, \lambda_i \in \mathbb{R}, i = 1, \dots, n\}.$$

A **Möbius n -sphere** is an extended n -hyperplane $P_n \cup \{\infty\}$ or a set of type $P_{n+1} \cap S(a, r)$ where P_{n+1} is an $(n+1)$ -hyperplane which contains a .

Proposition 2.1.3. [3] For any Möbius transformation, the image of a Möbius n -sphere is a Möbius n -sphere.

From now to the end of this subsection, we fix the basis $\{e_i\}_{i \in I}$ in \mathbb{H} and \mathbb{H} is the hyperplane which is orthogonal to e_1 . Each $x \in \mathbb{H}$ will be written $x = (x_1, \bar{x})$ with $\bar{x} \in \mathbb{H}$. We denote by $\mathbb{H}^+ = \{x \in \mathbb{H} : x_1 > 0\}$. Note that \mathbb{H}^+ is one side of the Möbius sphere $\widehat{\mathbb{H}}$.

We denote by $\mathfrak{M}(\mathbb{H})$ the group of all Möbius transformations of $\widehat{\mathbb{H}}$ such that $\mu(\mathbb{H}^+) = \mathbb{H}^+$. Then, from Proposition 2.1.2, for $\mu \in \mathfrak{M}(\mathbb{H})$, we have $\mu(\widehat{\mathbb{H}}) = (\widehat{\mathbb{H}})$.

The converse is also true:

If μ is a reflection of $\widehat{\mathbb{H}}$ on $P(a, t)$, let $\tilde{\mu}$ be the reflection in $\widehat{\mathbb{H}}$ on $\hat{P}((0, a), t)$ in $\widehat{\mathbb{H}}$.

If μ is a reflection of $\widehat{\mathbb{H}}$ on $S(a, r)$, let $\tilde{\mu}$ be the reflection in $\widehat{\mathbb{H}}$ on $\hat{S}((0, a), r)$ in $\widehat{\mathbb{H}}$.

If $\mu = \alpha\omega + a$ is a similitude of $\widehat{\mathbb{H}}$, let $\tilde{\mu} = \alpha\tilde{\omega} + (0, a)$ be the similitude in $\widehat{\mathbb{H}}$ where $\tilde{\omega}|_{\mathbb{H}} = \omega$ and $\tilde{\omega}(e_1) = e_1$.

In any case $\tilde{\mu}$ preserves \mathbb{H}^+ and \mathbb{H} . It follows that the group $\mathfrak{M}(\mathbb{H})$ of Möbius transformations of $\widehat{\mathbb{H}}$ is isomorphic to $\mathfrak{M}(\mathbb{H})$.

On the other hand, on \mathbb{H} , we consider the hyperbolic distance δ characterized by (cf [3])

$$\cosh \delta(x, y) = \sqrt{1 + |x|^2} \sqrt{1 + |y|^2} - \langle x, y \rangle \text{ and } \delta(x, y) \geq 0.$$

Definition 2.1.2. A bijection ϕ of \mathbb{H} is called a hyperbolic transformation if we have:

$$\forall x, y \in \mathbb{H}; \quad \delta(\phi(x), \phi(y)) = \delta(x, y).$$

Consider the diffeomorphism $h : \mathbb{H}^+ \rightarrow \mathbb{H}$ defined by

$$h(x_1, \bar{x}) = \left(\frac{|x|^2 - 1}{2x_1}, \frac{\bar{x}}{x_1} \right).$$

The link between hyperbolic transformations and Möbius transformations is given in the following result of [4].

Theorem 2.1.2. [4]

- (1) The group $\mathfrak{G}(\mathbb{H})$ of hyperbolic transformations of \mathbb{H} is the set $\{\phi = h \circ \mu \circ h^{-1} : \mu \in \mathfrak{M}(\mathbb{H})\}$.
- (2) Each map $\phi \in \mathfrak{G}(\mathbb{H})$ can be written as a similitude β or a product $\alpha \circ \rho_0 \circ \beta$ with α and β are of the form :
 - (i) $\alpha(x) = kx + v$ with $k > 0$, $v \in \mathbb{H}$;
 - (ii) $\beta(x) = k'\omega(x) + v'$ with $k' > 0$, $v' \in \mathbb{H}$, ω an orthogonal transformation of \mathbb{H} such that $\omega(v') = v'$.

From now on, we identify the groups $\mathfrak{M}(\mathbb{H})$ and $\mathfrak{G}(\mathbb{H})$.

Remark 2.1.1.

- (1) According to [4], the pair $(\mathbb{H}^+, \mathfrak{M}(\mathbb{H}))$ is called the Poincaré model of hyperbolic geometry. In fact, let $g_{\mathbb{H}^+} = \frac{1}{x^2}g$ be the conformal metric to the canonical Riemannian metric where g is induced by the inner product $\langle \cdot, \cdot \rangle$ on \mathbb{H} . Then the map $h : (\mathbb{H}^+, g_{\mathbb{H}^+}) \rightarrow (\mathbb{H}, \delta)$ is an isometry.
- (2) If \mathbb{H} is finite dimensional, each isometry is a bijection, but it is no longer true in general, if \mathbb{H} is infinite dimensional (see [3]).

2.2. Möbius transformations and the Lorentz group.

In this subsection, we consider $\mathcal{H} = \mathbb{R} \oplus \mathbb{H}$, the basis $\{e_i\}_{i \in I}$ is fixed and again \mathbb{H} is the orthogonal of e_1 in \mathbb{H} . We put on \mathcal{H} the following Lorentz product:

$$\langle (s, x), (t, y) \rangle_L = \langle x, y \rangle - st.$$

We then denote by $|\cdot|_L$ the associated pseudo-norm and by \mathcal{K} the *light cone* i.e. $\mathcal{K} = \{u = (s, x) \in \mathcal{H} : \langle u, u \rangle_L = 0\}$, and $\mathcal{K}^+ = \{u = (s, x) \in \mathcal{K} : s > 0\}$.

Definition 2.2.1. A bijection λ of \mathcal{H} is called a Lorentz transformation if we have

$$\forall u, v \in \mathcal{H}, |\lambda(u) - \lambda(v)|_L = |u - v|_L.$$

On the other hand, we consider the hyperboloid $\mathcal{H}_1 = \{u = (s, x) \in \mathcal{H} : |u|_L^2 = -1, \}$ and its "positive time like sheet" $\mathcal{H}_1^+ = \{u = (s, x) \in \mathcal{H}_1 : s > 0\}$. Let $g : \mathbb{H} \rightarrow \mathcal{H}_1^+$ be a bijection defined by $g(x) = (\sqrt{1 + |x|^2}, x)$.

The link between the Lorentz transformations of \mathcal{H} and the hyperbolic transformations of \mathbb{H} is given by the following result (cf [3]).

Theorem 2.2.1.

Given any hyperbolic transformation ϕ , there exists a unique Lorentz transformation $\lambda = \tau(\phi)$ such that

$$\lambda(0) = 0, \lambda(\mathcal{H}_1^+) = \mathcal{H}_1^+ \text{ and } \forall x \in \mathbb{H}, g(\phi(x)) = \lambda(g(x)).$$

Moreover the restriction to \mathcal{H}_1^+ of the Lorentz transformation $\tau(\phi)$ associated to ϕ is given by:

$$\tau(\phi)|_{\mathcal{H}_1^+} = g \circ \phi \circ g^{-1}.$$

According to this result, the Lorentz transformation of type $\lambda = \tau(\phi)$, where ϕ is a hyperbolic transformation, is then a continuous linear map which is called an *orthochronous Lorentz linear map*.

The set $SO(\mathbb{H}, 1)$ of linear Lorentz transformations λ such that $\lambda(\mathcal{K}^+) = \mathcal{K}^+$ is a subgroup of the group $O(\mathbb{H}, 1)$ of all linear Lorentz transformations and the set $SO_0(\mathbb{H}, 1)$ of orthochronous Lorentz linear maps is a subgroup of $SO(\mathbb{H}, 1)$. Moreover, according to Theorem 2.1.2, Remark 2.1.1 and Theorem 2.2.1 we have a natural isomorphism \mathcal{L} from the group of Möbius transformations $\mathfrak{M}(\mathbb{H})$ and the group $SO_0(\mathbb{H}, 1)$. More precisely we have:

$$\mathcal{L}(\lambda) = (g \circ h)^{-1} \circ \lambda|_{\mathcal{H}_1^+} \circ (g \circ h).$$

In fact, as \mathcal{H}_1^+ is the set $\{(s, x) \in \mathcal{H} \text{ such that } s = \sqrt{1 + |x|^2}\}$ and so \mathcal{H}_1^+ is a smooth hypersurface. In the restriction to \mathcal{H}_1^+ we have $\langle (s, x), (t, y) \rangle_L = \langle x, y \rangle - \sqrt{1 + |x|^2} \sqrt{1 + |y|^2}$. Therefore, in the restriction to \mathcal{H}_1^+ $\cosh \delta(s, x), (t, y) = -\langle (s, x), (t, y) \rangle_L$ defines a hyperbolic distance and the map $g(x) = (\sqrt{1 + |x|^2}, x)$ is a diffeomorphism from \mathbb{H} to \mathcal{H}_1^+ which is an isometry. According to Theorem 2.1.2 and Theorem 2.2.1 we get an natural identification of the group $\mathfrak{M}(\mathbb{H})$ and the group of the restriction to \mathcal{H}_1^+ of elements of $SO_0(\mathbb{H}, 1)$.

We end this subsection by recalling a characterization of the group $SO(\mathbb{H}, 1)$ and its Lie algebra (cf [5] or [11]). We adopt the presentation of [11].

According to the decomposition $\mathcal{H} = \mathbb{R} \oplus \mathbb{H}$, let p_1 be (resp. p_2) the natural projection of \mathcal{H} onto \mathbb{R} (resp. \mathbb{H}). It follows that each continuous linear map A of \mathcal{H} in a obvious matrix form

$$(2.2.1) \quad \begin{pmatrix} c & [v]^* \\ [u] & B \end{pmatrix}$$

where $c = p_1(A(1, 0))$, $B = p_2 \circ A|_{\mathbb{H}}$ and u (resp v) is an element of \mathbb{H} such that $p_2 \circ A(1, 0) = u$ and $[u](s) = su$ (resp. $p_1 \circ A(0, x) = \langle v, x \rangle$ and $[v]^*(x) = \langle v, x \rangle$).

Now, let J be the continuous endomorphism of \mathcal{H} defined by $J(s, x) = (-s, x)$. Given a continuous endomorphism A of \mathcal{H} , the pseudo-adjoint $A^\#$ is the continuous endomorphism characterized by

$$\langle Au, v \rangle_L = \langle u, A^\#v \rangle_L \text{ for any } u, v \in \mathcal{H}.$$

Thus, A belongs to $O(\mathbb{H}, 1)$ (resp. $SO(\mathbb{H}, 1)$) if and only if $A^\#A = Id$ (resp. $A^\#A = Id$ and $A^\# \in SO(\mathbb{H}, 1)$).

According to the matrix form (2.2.1), $A^\#$ has a matrix form of type

$$(2.2.2) \quad \begin{pmatrix} c & -[u]^* \\ -[v] & B^* \end{pmatrix}$$

where B^* is the adjoint endomorphism (of \mathbb{H}) of B . Then A belongs to $O(\mathbb{H}, 1)$ if and only if

$$(2.2.3) \quad \begin{pmatrix} c & -[u]^* \\ -[v] & B^* \end{pmatrix} \begin{pmatrix} c & [v]^* \\ [u] & B \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & Id \end{pmatrix}$$

where Id is the identity in \mathbb{H} . Moreover, $A \in O(\mathbb{H}, 1)$ belongs to $SO(\mathbb{H}, 1)$ if and only if $c > 0$ (see [11]).

The following result is classical in the finite dimensional case and in the infinite dimensional case it is more or less included in [3] or [11]

Proposition 2.2.1.

Let $A \in O(\mathbb{H}, 1)$, there exists, $v \in \mathbb{H}$ with $v \neq 0$, such that A has the following decomposition:

$$(2.2.4) \quad A = PT$$

where $P = \begin{pmatrix} \varepsilon & 0 \\ 0 & Q \end{pmatrix}$ and $Q^{-1} = Q^*$, $\varepsilon = \pm 1$ and T is such that:

if H_v is the orthogonal of $\mathbb{R}.v$ in \mathbb{H} then $T|_{H_v} = Id_{H_v}$ and $T(\mathbb{R} \oplus \mathbb{R}.v) = \mathbb{R} \oplus \mathbb{R}.v$. Moreover, there exists $\alpha \geq 0$ such that the eigenvalues of $T|_{\mathbb{R} \oplus \mathbb{R}.v}$ are e^α and $e^{-\alpha}$ with associated eigenvectors $(\frac{v}{|v|}, 1)$ and $(\frac{v}{|v|}, -1)$ respectively.

Note that, in the previous decomposition, T is called a *Lorentz boost* and it is characterized by $u \in \mathbb{H}$ and $\alpha > 0$ so it will denoted by $B_{u, \alpha}$. Moreover according to Theorem 2.2.1, T is associated to a *hyperbolic translation* generated by v . (cf [3]). Note that if $\{u_i\}_{i \in I, i > 1}$ is an orthonormal basis of H_v , let Q be the linear isometry in \mathbb{H} such that $Q(e_1) = \frac{v}{|v|}$ and $Q(e_i) = u_i, i \in I, i > 1$. Then we have:

$$(2.2.5) \quad B_{u, \alpha} = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 \\ \sinh \alpha & \cosh \alpha & 0 \\ 0 & 0 & Id_{H_v} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q^* \end{pmatrix}.$$

Thus we get the following corollary (see also [3]):

Corollary 2.2.1.

For any $A \in O(\mathbb{H}, 1)$ there exists Q and Q' in $SO(\mathbb{H})$ and $\alpha > 0$ such that

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & Q' \end{pmatrix} \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 \\ \sinh \alpha & \cosh \alpha & 0 \\ 0 & 0 & Id_{\mathbb{H}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q^* \end{pmatrix}.$$

Remark 2.2.1. According to [3] and our identifications, any boost is a hyperbolic translation. Moreover, as in the finite dimension, in the metric space (H^+, g_{H^+}) (cf Remark 2.1.1 (1)), any boost $B_{e_1, \alpha}$ corresponds to the homothety $x \rightarrow e^\alpha . x$ in H^+ and so to the Möbius transformation $x \rightarrow e^\alpha . x$ in \hat{H} .

The proof of Proposition 2.2.1 is an adaptation to our context of comparable result of the finite dimensional case in [5]

Proof. According to (2.2.1), (2.2.2) and (2.2.3), we get:

$$B^*B = Id_{\mathbb{H}} + [v][v]^* \quad [u]^*[u] = c^2 - 1 \quad [u]^*B = c[v]^* \quad B^*u = cv$$

and also

$$BB^* = Id_{\mathbb{H}} + [u][u]^* \quad [v]^*[v] = c^2 - 1 \quad [v]^*B = c[u]^* \quad Bv = cu.$$

On one hand, we get as $[v]^*[v] = |v|^2$ so $c^2 = 1 + |v|^2$ and $c^2 = 1 + |u|^2$ in particular $u \neq 0$. On the other hand the kernel of $[v][v]^*$ is the orthogonal H_v of $\mathbb{R}.v$ in \mathbb{H} . It follows that the restriction of $[v][v]^*$ to H_v is zero and the restriction $[v][v]^*$ to $\mathbb{R}.v$ is such that $[v][v]^*(v) = |v|^2.v = (c^2 - 1)v$. We deduce that $(Id_{\mathbb{H}} + [v][v]^*)|_{H_v} = Id_{H_v}$ and v is an eigenvector of $Id_{\mathbb{H}} + [v][v]^*$ with eigenvalue c^2 of multiplicity 1. From the polar decomposition theorem in Hilbert space, there exists a linear isometry Q of \mathbb{H} and a self-adjoint positive definite operator S (on \mathbb{H}) such that $B = QS$. Moreover, we have $B^*B = S^2$ and so, $S|_{H_v} = Id_{H_v}$ and $S(v) = \pm cv$. We may assume that this eigenvalue

c is positive after changing eventually c into $-c$. Therefore we have $S(v) = cv$.

Assume at first that $c > 0$. Since $Bv = cu$, then $QS(v) = cQ(v) = cu$ and so $Q(v) = u$. We get

$$(2.2.6) \quad \begin{pmatrix} c & [v]^* \\ [u] & B \end{pmatrix} = \begin{pmatrix} c & [v]^* \\ Qv & QS \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} c & [v]^* \\ [v] & S \end{pmatrix}$$

with $c = \sqrt{|v|^2 + 1}$ and $\varepsilon = 1$. We set $T = \begin{pmatrix} c & [v]^* \\ [v] & S \end{pmatrix}$.

If $c < 0$ by an analogue argument we get a decomposition as (2.2.6) but with $\varepsilon = -1$.

Now, the restriction of T to \mathbb{H}_v is $Id_{\mathbb{H}_v}$ and, (in \mathcal{H}), $T(\mathbb{R} \oplus \mathbb{R}.v) = \mathbb{R} \oplus \mathbb{R}.v$. By similar arguments used in the proof of Proposition 2.4 of [5] we complete the proof. \square

In the sequence we denote by $\sqrt{Id_{\mathbb{H}} + [v][v]^*}$ the operator S and so we have

$$(2.2.7) \quad T = \begin{pmatrix} c & [v]^* \\ [v] & \sqrt{Id_{\mathbb{H}} + [v][v]^*} \end{pmatrix} \text{ and } A = \begin{pmatrix} \varepsilon & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} c & [v]^* \\ [v] & \sqrt{Id_{\mathbb{H}} + [v][v]^*} \end{pmatrix}.$$

Assume now that $I = \{1, \dots, n\}$. According to Proposition 2.2.1 (see [5]) any matrix $A \in O(n, 1)$ can be written as a product of matrices of the form

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} c & [v]^* \\ [v] & \sqrt{Id_n + [v].[v]^*} \end{pmatrix}$$

where Q belongs to $O(n)$, $[v]$ is a vector column of \mathbb{H} and $c = \sqrt{|v|^2 + 1}$ and $\varepsilon = \pm 1$.

Thus, the Lie group $O(n, 1)$ has 4 connected components, according to the previous decomposition, we have $\det Q = \pm 1$ and $\varepsilon = \pm 1$. The group of Lorentz transformations is $SO(n, 1)$ which is the group corresponding to $\det Q = \varepsilon = \pm 1$. According to the previous Proposition and Theorem 2.1.2, the group $\mathfrak{M}(\mathbb{H})$ is isomorphic to $SO(n, 1)$, and so the group $\mathfrak{M}^+(\mathbb{H})$ which preserves the orientation is isomorphic to the connected components of the Identity in $SO(n, 1)$, that is the subgroup $SO_0(n, 1)$ corresponding to the case $\det Q = \varepsilon = 1$.

On the other hand (see [5] for instance), the Lie algebra $\mathfrak{so}(n, 1)$ of $SO_0(n, 1)$ is the set of matrices of the form

$$\begin{pmatrix} 0 & [u]^* \\ [u] & B \end{pmatrix}$$

where B is a square matrix of dimension n such that $B^* = -B$. Therefore we have a natural decomposition

$$\mathfrak{so}(n, 1) = \mathfrak{h}_n \oplus \mathfrak{s}_n$$

where

$$\mathfrak{h}_n = \left\{ \begin{pmatrix} 0 & [u]^* \\ [u] & 0 \end{pmatrix} \text{ where } [u] \text{ vector column } \in \mathbb{R}^n \right\}$$

$$\mathfrak{s}_n = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} B^* = -B \right\}.$$

The vector space \mathfrak{h}_n is generated by $U_i = \begin{pmatrix} 0 & [e_i]^* \\ [e_i] & 0 \end{pmatrix}$ for $i = 1, \dots, n$ and \mathfrak{s}_n is a Lie subalgebra of $\mathfrak{so}(n, 1)$ generated by $\Omega_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \omega_{ij} \end{pmatrix}$ $1 \leq i < j \leq n$, where ω_{ij} is the matrix with the term of index ij (resp. ji) is 1 (resp. -1) and the other terms are 0.

Remark 2.2.2.

- (1) When $I = \mathbb{N}$, the group $O(\mathbb{H}, 1)$ is a Lie subgroup of the group $GL(\mathcal{H})$ of continuous automorphism of \mathcal{H} . However, this group has only two connected components and in particular $SO_0(\mathbb{H}, 1) = SO(\mathbb{H}, 1)$. On the other hand, in the decomposition 2.2.7, T belongs to $SO(\mathbb{H}, 1)$ so A in (2.2.7) belongs to $SO(\mathbb{H}, 1)$ if and only if $\varepsilon = 1$.

The Lie algebra $\mathfrak{so}(\mathbb{H}, 1)$ of $SO(\mathbb{H}, 1)$ has also a decomposition of type $\mathfrak{h} \oplus \mathfrak{s}$ where \mathfrak{h} is the set of endomorphism of type $\begin{pmatrix} 0 & [u]^* \\ [u] & 0 \end{pmatrix}$ where $u \in \mathbb{H}$ and \mathfrak{s} is a Lie algebra that is, the set of endomorphisms of type $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ where $B^* = -B$.

In fact \mathfrak{s} is isomorphic to the Lie algebra of the group of linear isometry of \mathbb{H} (cf [11]).

- (2) Consider the exponential map $\text{Exp} : \mathfrak{so}(\mathbb{H}, 1) \rightarrow SO(\mathbb{H}, 1)$. When $I = \{1, \dots, n\}$, each boost T can be written as $\text{Exp}U$, for some $U \in \mathfrak{h}_n$ (cf [5] for instance). On the other hand, each $P \in SO(n)$ can also be written as $\text{Exp}\Omega$ for some Ω of the Lie algebra of $SO(n)$. This implies that each element of $SO(n, 1)$ can be written as $\text{Exp}\Omega\text{Exp}(U)$ for some $\Omega \in \mathfrak{s}_n$ and $U \in \mathfrak{h}_n$. Unfortunately Ω and U do not commute and so $\text{Exp}(\Omega)\text{Exp}(U) \neq \text{Exp}(\Omega + U)$ and we do not get the surjectivity property of Exp . However, $\text{Exp} : \mathfrak{so}(n, 1) \rightarrow SO_0(n, 1)$ is surjective (see [5] section 4.5).

3. THE HILBERT-SCHMIDT MÖBIUS GROUP OF THE UNIT SPHERE OF \mathbb{H}

3.1. Hilbert-Schmidt group of orthochronous Lorentz transformations.

Given a Hilbert space \mathbb{H} , we first recall results of [7], about some particular Lie sub-algebras of $L(\mathbb{H})$ of the Lie algebra $L(\mathbb{H})$ of bounded operators on \mathbb{H} .

We consider a family $(G_n)_{n \in \mathbb{N}}$ of connected finite dimensional Lie subgroups of $GL(\mathbb{H})$ such that

$$G_1 \subset G_2 \subset \dots \subset G_n \subset \dots \subset GL(\mathbb{H})$$

where $GL(\mathbb{H})$ denote the group of invertible elements of $L(\mathbb{H})$.

Let \mathfrak{g}_n be the Lie algebra of G_n and $\mathfrak{g} = \bigcup_{n \in \mathbb{N}} \mathfrak{g}_n$. Then \mathfrak{g} is a Lie algebra.

Assumptions 3.1.1. *There exists a subspace \mathfrak{g}_∞ in $L(\mathbb{H})$ which contains \mathfrak{g} and such that we can extend the inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} to an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g}_∞ , which is complete for the associated norm $\|\cdot\|$ and such that \mathfrak{g} is dense in \mathfrak{g}_∞ . Moreover, we assume that \mathfrak{g}_∞ is closed under Lie bracket of $L(\mathbb{H})$ and there exists a constant $C > 0$ such that*

$$(3.1.1) \quad \|[A, B]\| \leq C\|A\|\|B\|.$$

Let $C_{\mathfrak{g}}^1$ be the set of piecewise C^1 paths γ from $[0, 1]$ to the Banach manifold $GL(\mathbb{H})$ such that

$$\gamma' = \gamma^{-1} \circ \dot{\gamma} \text{ belongs to } \mathfrak{g}_\infty \text{ and } \gamma' \text{ is piecewise continuous for the norm } \|\cdot\| \text{ (on } \mathfrak{g}_\infty).$$

On $GL(\mathbb{H})$ we define:

$$d(A, B) = \inf \left\{ \int_0^1 |\gamma'(s)| ds : \gamma \in C_{\mathfrak{g}}^1 \text{ such that } \gamma(0) = A, \gamma(1) = B \right\}$$

$$d(A, B) = \infty \text{ if there is no } \gamma \in C_{\mathfrak{g}}^1 \text{ such that } \gamma(0) = A, \gamma(1) = B.$$

Theorem 3.1.1. [7] *Under the previous assumptions we have*

- (1) *Let $G_\infty = \{A \in GL(\mathbb{H}) : d(A, Id_{\mathbb{H}}) < \infty\}$. Then G_∞ is a subgroup of $GL(\mathbb{H})$ and d is a distance on this set which is left invariant.*
- (2) *For the topology associated to d the group G_∞ is closed, and the group $G = \bigcup_{n \in \mathbb{N}} G_n$ is dense in G_∞ .*
- (3) *Let d_n be the distance associated to the norm $\|\cdot\|$ on \mathfrak{g}_n . Then the distance $d_\infty = \inf_{n \in \mathbb{N}} d_n$ on G coincides with the restriction of d .*
- (4) *The exponential map $\text{Exp} : \mathfrak{g}_\infty \rightarrow G_\infty$ is a local diffeomorphism around 0 in \mathfrak{g}_∞ .*

In particular, G_∞ is a Lie group modeled on the Hilbert space \mathfrak{g}_∞ .

The group G_∞ is called a **Cameron-Martin group** (cf [7]).

From now to the end of this subsection, we fix a Hilbert basis $\{e_i\}_{i \in \mathbb{N} \setminus \{0\}}$ of \mathbb{H} and $\mathcal{H} = \mathbb{R} \oplus \mathbb{H}$ is now equipped with the Hilbert inner product $\langle (s, x), (t, y) \rangle = st + \langle x, y \rangle$.

We can identify \mathcal{H} with $l^2(\mathbb{N})$. Let $L_{HS}(\mathcal{H})$ be the subspace of Hilbert-Schmidt operators of \mathcal{H} , that is

$$L_{HS}(\mathcal{H}) = \{A \in L(\mathcal{H}) \text{ such that } \sum_{i \in \mathbb{N}} |Ae_i|^2 < \infty\}.$$

Recall that on $L_{HS}(\mathcal{H})$ we have an inner product

$$\langle A, B \rangle_{HS} = \sum_{i \in \mathbb{N}} \langle Ae_i, Be_i \rangle$$

and the associated norm is

$$|A|_{HS} = \left(\sum_{i \in \mathbb{N}} |Ae_i|^2 \right)^{\frac{1}{2}}.$$

Note that $L_{HS}(\mathcal{H})$ is then a Hilbert space.

We can consider each operator $A \in L_{HS}(\mathcal{H})$ as an infinite matrix $A = (a_{ij})_{i,j \in \mathbb{N}}$ such that $\sum_{i,j \in \mathbb{N}} |a_{ij}|^2 < \infty$.

Therefore, if e_{ij} denote the infinite matrix defined by:

1 at the ij th place and 0 at all other places,

we get an orthonormal basis $\{e_{ij}\}$ of $L_{HS}(\mathcal{H})$ (relative to the inner product $\langle \cdot, \cdot \rangle_{HS}$). Note that $L_{HS}(\mathcal{H})$ is a Banach algebra (without unit) for the norm $\|\cdot\|_{HS}$ (cf [15]). In the Banach Lie group $GL(\mathcal{H})$ of invertible bounded operators, the general Hilbert-Schmidt group is

$$GL_{HS}(\mathcal{H}) = \{U \in L(\mathcal{H}) \text{ such that } Id_{\mathcal{H}} - U \in L_{HS}(\mathcal{H})\}.$$

On the other hand, denote by \mathbb{H}_n the vector space generated by $\{e_1, \dots, e_n\}$, and \mathcal{H}_n the vector space $\mathbb{R} \oplus \mathbb{H}_n$. Now, we can identify $L(\mathcal{H}_n)$ with the set

$$L_n(\mathcal{H}) = \{A \in L_{HS}(\mathcal{H}) : \mathcal{H}_n^\perp \subset \ker A \text{ and } \text{Im} A \subset \mathcal{H}_n\}.$$

Since we have $\mathcal{H}_n \subset \mathcal{H}_{n+1}$, we have $\mathcal{H}_{n+1}^\perp \subset \mathcal{H}_n^\perp$ so, if $A \in L_n(\mathcal{H})$ then A belongs to $L_{n+1}(\mathcal{H})$. In this way we obtain an ascending family:

$$(3.1.2) \quad L_1(\mathcal{H}) \subset L_2(\mathcal{H}) \subset \dots \subset L_n(\mathcal{H}) \subset \dots \subset L_{HS}(\mathcal{H}) \subset L(\mathcal{H}).$$

In the same way, we can identify $GL(\mathcal{H}_n)$ with the set

$$GL_n(\mathcal{H}) = \left\{ A \in GL_{HS}(\mathcal{H}) \text{ of type } \begin{pmatrix} Id_{\mathcal{H}_n^\perp} & 0 \\ 0 & \bar{A} \end{pmatrix} \bar{A} \in GL(\mathcal{H}_n) \right\}$$

and by the similar arguments, we have also an ascending family

$$(3.1.3) \quad GL_1(\mathcal{H}) \subset GL_2(\mathcal{H}) \subset \dots \subset GL_n(\mathcal{H}) \subset \dots \subset GL_{HS}(\mathcal{H}) \subset GL(\mathcal{H}).$$

If A belongs to $GL_{HS}(\mathcal{H})$ then the determinant of A is well defined and $\det(A) \neq 0$. Moreover, according to the previous construction, any $A \in GL_{HS}(\mathcal{H})$ induces a natural endomorphism $A_n \in GL_n(\mathcal{H})$. We have then (cf [17])

$$(3.1.4) \quad \det(A) = \lim_{n \rightarrow \infty} \det(A_n)$$

Now, modulo the previous identification and according to the end of subsection 2.2, the family $(\mathfrak{so}(n, 1))_{n \in \mathbb{N}}$ becomes a family of Lie subalgebras of $L_{HS}(\mathcal{H})$ and the family of Lie groups $(SO_0(n, 1))_{n \in \mathbb{N}}$ becomes a family of ascending Lie subgroups of $GL(\mathcal{H})$ whose Lie algebras is the family $(\mathfrak{so}(n, 1))_{n \in \mathbb{N}}$.

According to the end of subsection 2.2 and the previous notations, let $U_i \in L_{HS}(\mathcal{H})$ such that $U_i = e_{0i} + e_{i0}$ for $i \in \mathbb{N} \setminus \{0\}$ and $\Omega_{ij} = e_{ij} - e_{ji}$ for $0 < i < j$, $i, j \in \mathbb{N}$. We denote by $\mathfrak{h}_\infty \subset L_{HS}(\mathcal{H})$ the Hilbert space generated by $\{U_i\}_{i \in \mathbb{N} \setminus \{0\}}$ and $\mathfrak{s}_\infty \subset L_{HS}(\mathcal{H})$ the Hilbert space generated $\{\Omega_{ij}\}_{0 < i < j, i, j \in \mathbb{N}}$. We set $\mathfrak{g}_\infty = \mathfrak{h}_\infty \oplus \mathfrak{s}_\infty$, according to the identification of $L(\mathcal{H}_n)$ with $L_n(\mathcal{H})$, we can consider $\mathfrak{so}(n, 1)$ as a subspace of \mathfrak{g}_∞ .

From Theorem 3.1.1 we will deduce the following:

Proposition 3.1.1.

- (1) The vector space \mathfrak{g}_∞ is the closure of $\mathfrak{g} = \bigcup_{n \in \mathbb{N}} \mathfrak{so}(n, 1)$ in $L_{HS}(\mathcal{H})$. Moreover \mathfrak{g}_∞ is Lie subalgebra of $L_{HS}(\mathcal{H})$ which satisfies the assumption 3.1.1.
- (2) The Cameron-Martin group G_∞ associated to the ascending sequence $(SO_0(n, 1))_{n \in \mathbb{N}}$ in $L(\mathcal{H})$ is a Lie subgroup of $GL_{HS}(\mathcal{H})$ and $\bigcup_{n \in \mathbb{N}} SO_0(n, 1)$ is dense in G_∞ . Moreover, \mathfrak{g}_∞ is the Lie algebra of G_∞ .
- (3) Each element A of G_∞ can be written as $A = PT$ where T is a boost and $P = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$ with $Q^{-1} = Q^*$ and $\det(Q) = 1$. In particular, $SO(\mathbb{H}, 1) \cap GL_{HS}(\mathcal{H})$ has two connected components and G_∞ is the connected component of $Id_{\mathcal{H}}$.
- (4) The map $\text{Exp} : \mathfrak{g}_\infty \rightarrow G_\infty$ is a surjective local diffeomorphism around $0 \in \mathfrak{g}_\infty$.

Remark 3.1.1.

Note that the Lie group $SO(\mathbb{H}, 1)$ is connected (cf Remark 2.2.2 part (1)) while $SO(\mathbb{H}, 1) \cap GL_{HS}(\mathcal{H})$ has two connected components.

Definition 3.1.1. The sub-group G_∞ of $SO(\mathbb{H}, 1)$ built in Proposition 3.1.1 is called the Hilbert-Schmidt orthochronous Lorentz group and will be denoted $SO_{HS}(\mathbb{H}, 1)$. The corresponding Lie algebra \mathfrak{g}_∞ will be denoted $\mathfrak{so}_{HS}(\mathbb{H}, 1)$.

In the remaining part of the article, we simply denote by \mathfrak{h} (resp. \mathfrak{s}) each subspace $\mathfrak{h}_\infty \subset \mathfrak{g}_\infty$ (resp. $\mathfrak{s}_\infty \subset \mathfrak{g}_\infty$) and so we get

$$(3.1.5) \quad \mathfrak{so}_{HS}(\mathbb{H}, 1) = \mathfrak{h} \oplus \mathfrak{s}.$$

If we now consider the natural isomorphism $\mathcal{L} : SO(\mathbb{H}, 1) \rightarrow \mathfrak{M}(\mathbb{H})$ (cf subsection 2.2), we get a subgroup $\mathfrak{M}_{HS}(\mathbb{H}) = \mathcal{L}(SO_{HS}(\mathbb{H}, 1))$ of $\mathfrak{M}(\mathbb{H})$. In this way, $\mathfrak{M}_{HS}(\mathbb{H})$ can be provided with a Lie group structure and its Lie algebra $\mathfrak{m}_{HS}(\mathbb{H})$ is isomorphic to $\mathfrak{so}_{HS}(\mathbb{H}, 1)$.

Definition 3.1.2. *The group $\mathfrak{M}_{HS}(\mathbb{H})$ is called the Hilbert-Schmidt group of Möbius transformations of \mathbb{H} .*

In finite dimension, in [6], the authors gives a complete description of the map $\text{Exp} : \mathfrak{so}(n) \rightarrow SO(n)$. Using similar results in an infinite dimensional Hilbert space context, we obtain:

Theorem 3.1.2.

Consider $\text{Exp} : \mathfrak{so}_{HS}(\mathbb{H}, 1) \rightarrow SO_{HS}(\mathbb{H}, 1)$ and fix some $A = PT \in SO_{HS}(\mathbb{H}, 1)$. According to (3.1.5), there exists $U \in \mathfrak{h}$, a family $\{B_j\}_{j \in J} \subset \mathfrak{s}$ with $J \subset \mathbb{N}$ of finite rank and a non increasing sequence $(\theta_j)_{j \in J}$ of real numbers with $0 < \theta_j \leq \pi$ with the following properties

- (i) $[B_k, B_j] = 0$ for $k \neq j$,
- (ii) $A = \prod_{j \in J} \text{Exp}(\theta_j B_j) \text{Exp } U$.

As the proof of the Theorem 3.1.2, is technical and has no direct relation with the context of Möbius transformation, we will give its proof in Appendix 5.1.

Proof of Proposition 3.1.1.

According to Theorem 3.1.1, we have only to prove that \mathfrak{g}_∞ satisfies the assumption 3.1.1. At first, by construction, as $\mathfrak{so}(n, 1)$ is a subset of $L_n(\mathcal{H})$, for each $n \in \mathbb{N} \setminus \{0\}$, $\mathfrak{so}(n, 1)$ is generated by $\{U_i\}_{1 \leq i \leq n}$, $\{\Omega_{ij}\}_{1 < i < j \leq n}$ so, $\mathfrak{g} = \bigcup_{n \in \mathbb{N}} \mathfrak{so}(n, 1)$ is dense in \mathfrak{g}_∞ . Also by construction, the natural inner product on $L_n(\mathcal{H})$ which is isometric to the canonical inner product of $L(\mathcal{H}_n)$ so that $\{e_{ij}\}_{i,j \in \mathbb{N} \setminus \{0\}}$ is the canonical orthonormal basis. It follows that \mathfrak{g}_∞ is a closed subspace of $L_{HS}(\mathcal{H})$, which is provided with an inner product extends the inner product on each $\mathfrak{so}(n, 1)$. On the other hand, by an elementary calculation, according to the Lie bracket $[A, B] = AB - BA$ on $L(\mathcal{H})$ we have the following relations:

$$(3.1.6) \quad [U_i, U_j] = \Omega_{jk}, \quad [U_i, \Omega_{jl}] = \delta_{ij}U_l - \delta_{il}U_j, \quad [\Omega_{ij}, \Omega_{kl}] = \delta_{il}\Omega_{jk} + \delta_{jk}\Omega_{il} - \delta_{ik}\Omega_{jl} - \delta_{jl}\Omega_{ik}.$$

It follows that \mathfrak{g}_∞ is closed under the Lie bracket of $L_{HS}(\mathcal{H})$. It remains to show that relation (3.1.1) is satisfied for any A and B in \mathfrak{g}_∞ . According to (3.1.6), the definition of U_i and Ω_{ij} , and the fact that $\{e_{ij}\}_{i,j \in \mathbb{N}}$ is an orthonormal basis in $L_{HS}(\mathcal{H})$ we have the following majorations:

$$(3.1.7) \quad \|[U_i, U_j]\|_{HS} \leq 2, \quad \|[U_i, \Omega_{jk}]\|_{HS} \leq 4, \quad \|[\Omega_{ij}, \Omega_{kl}]\|_{HS} \leq 8.$$

Now, any $A \in \mathfrak{g}_\infty$ can be written (using Einstein convention):

$$A = u^i U_i + a^{ij} \Omega_{ij},$$

so $|A|_{HS}^2 = 2(\sum_{i \in \mathbb{N}} (u^i)^2 + \sum_{0 < i < j, i, j \in \mathbb{N}} (a^{ij})^2)$. According to the bi-linearity of $[\cdot, \cdot]$, relations (3.1.6) and (3.1.7) easily get a relation of type (3.1.1) for the Lie bracket on \mathfrak{g}_∞ .

The other properties in (1) and (2) are direct consequences of Theorem 3.1.1.

Any $M \in GL_{HS}(\mathcal{H})$ induces a natural element $M_n \in GL_n(\mathcal{H})$, and of course, $GL_{HS}(\mathcal{H})$ is the Cameron-Martin group associated to the ascending family (3.1.3). In particular, according to the notations of Theorem 3.1.1, we have:

$$(3.1.8) \quad \lim_{n \rightarrow \infty} d_\infty(M, M_n) = 0.$$

Now, Let $A \in G_\infty$. As $G_\infty \subset SO(\mathbb{H}, 1)$, according to Proposition 2.2.1, we can write $A = PT$ where T is a boost and $P = \begin{pmatrix} \varepsilon & 0 \\ 0 & Q \end{pmatrix}$ with $Q^{-1} = Q^*$. With the previous convention, for each n , we have $A_n = P_n T_n$ where T_n is a boost in \mathcal{H}_n and $P_n = \begin{pmatrix} \varepsilon & 0 \\ 0 & Q_n \end{pmatrix}$ with $(Q_n)^{-1} = (Q_n)^*$. By construction of G_∞ , A_n belongs to $SO_0(\mathbb{H}_n, 1)$ so $\varepsilon = 1$ and $\det(Q_n) = 1$. From (3.1.8), in P we must have $\varepsilon = 1$ and $\det(Q) = 1$. The same arguments applied to $A \in SO(\mathbb{H}, 1) \cap GL_{HS}(\mathcal{H})$ implies that $A = PT$ with $P = \begin{pmatrix} \varepsilon & 0 \\ 0 & Q \end{pmatrix}$ and $\det(Q) = \varepsilon = \pm 1$. This ends Part (3).

As $\text{Exp} : \mathfrak{g}_\infty \rightarrow G_\infty$ is a smooth map, Part (4) is then a consequence of Point (2) of Remark 2.2.2 and the construction of G_∞ . □

3.2. Hilbert-Schmidt Möbius group of the unit sphere of \mathbb{H} .

Given a Hilbert basis $\{e_i\}_{i \in I}$ we again denote by \mathbf{H} the orthogonal of e_1 . Consider any $v \in \mathbb{H}$ with $v \neq 0$ and \mathbf{H}_v the orthogonal of $\mathbb{R} \cdot v$ in \mathbb{H} . If e_1 and v are linearly independent, after changing v into $-v$ if necessary, we may assume that $\langle v, e_1 \rangle = v_1 \geq 0$ so v belongs to $\mathbf{H}^+ = \{x, : x_1 \geq 0\}$. If we set $e = \frac{v}{|v|}$, we have an orthogonal isometry R_v such that $R_v(e) = e_1$ and then $R_v(\mathbf{H}_v) = \mathbf{H}$. We get an isomorphism from the group $\mathfrak{M}(\mathbf{H})$ to the group $\mathfrak{M}(\mathbf{H}_v)$ of Möbius transformations of $\widehat{\mathbf{H}}_v = \mathbf{H}_v \cup \{\infty\}$. Then we identify these groups.

In \mathbb{H} we consider the unit sphere $\mathbb{S}_{\mathbb{H}} = \{z \in \mathbb{H}, |z| = 1\}$ and the point $N = (1, \bar{0})$. The stereographic projection (cf [3]) is the map:

$$\Pi : \mathbb{S}_{\mathbb{H}} \setminus \{N\} \longrightarrow \mathbf{H} \quad (x_1, \bar{x}) \longmapsto \frac{\bar{x}}{1 - x_1}.$$

We can extend Π to $\mathbb{S}_{\mathbb{H}}$ into $\widehat{\mathbf{H}}$ by setting $\Pi(1, \bar{0}) = \infty$. Then Π becomes an homeomorphism from $\mathbb{S}_{\mathbb{H}}$ to $\widehat{\mathbf{H}}$, whose inverse is the map

$$\bar{x} \longmapsto \left(\frac{|\bar{x}|^2 - 1}{|\bar{x}|^2 + 1}, \frac{2\bar{x}}{|\bar{x}|^2 + 1} \right) \text{ and } \infty \longmapsto N.$$

Definition 3.2.1.

A diffeomorphism ϕ of $\mathbb{S}_{\mathbb{H}}$ is called a Möbius transformation of $\mathbb{S}_{\mathbb{H}}$ if $\Pi \circ \phi \circ \Pi^{-1}$ belongs to $\mathfrak{M}(\mathbf{H})$.

The group of Möbius transformations of $\mathbb{S}_{\mathbb{H}}$ is denoted $\mathfrak{M}(\mathbb{S}_{\mathbb{H}})$. Thus, modulo a choice of a Hilbert basis we get an isomorphism $\mathcal{P} : \mathfrak{M}(\mathbb{S}_{\mathbb{H}}) \rightarrow \mathfrak{M}_{HS}(\mathbf{H})$. Let $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ be the subgroup associated $\mathfrak{M}_{HS}(\mathbf{H}) = \{\mu|_{\mathbb{H}}, \mu \in \mathfrak{M}_{HS}(\mathbb{H})\}$ via the isomorphism \mathcal{P} . This group will be called the *Hilbert-Schmidt Möbius group* of $\mathbb{S}_{\mathbb{H}}$. The Lie algebra $\mathfrak{m}_{HS}(\mathbb{S}_{\mathbb{H}})$ of this group is then isomorphic to \mathfrak{g}_{∞} .

Consider $v \in \mathbb{H}$ with $v \neq 0$. There exists $R_v \in O(\mathbb{H})$ such that $R_v(\mathbf{H}_v) = \mathbf{H}$ (see the beginning of this subsection) and so we have $\Pi_v = R_v^* \circ \Pi$, R_v^* is the adjoint of R_v . It follows that ϕ belongs to $\mathfrak{M}(\mathbb{S}_{\mathbb{H}})$ if and only if $\Pi_v \circ \phi \circ \Pi_v^{-1}$ belongs to $\mathfrak{M}(\mathbf{H}_v)$ and then our definition of $\mathfrak{M}(\mathbb{S}_{\mathbb{H}})$ is independent of the choice of the basis $\{e_i\}_{i \in I}$ of \mathbb{H} .

Now, the unit sphere $\mathbb{S}_{\mathbb{H}}$ is a Hilbert submanifold of \mathbb{H} , and the tangent space $T_z \mathbb{S}_{\mathbb{H}}$ at $z \in \mathbb{S}_{\mathbb{H}}$ can be identified with the hyperplane \mathbf{H}_v . We denote by $g_{\mathbb{S}_{\mathbb{H}}}$ the Riemannian metric on $T\mathbb{S}_{\mathbb{H}}$ induced by \langle, \rangle . Let φ_v be the function on $\mathbb{S}_{\mathbb{H}}$ defined by $\varphi_v(x) = \langle \frac{v}{|v|}, x \rangle$. The gradient of φ_v (relative to the Riemannian metric $g_{\mathbb{S}_{\mathbb{H}}}$) is the vector field on $\mathbb{S}_{\mathbb{H}}$ defined by:

$$(3.2.1) \quad \text{grad}(\varphi_v)(x) = \frac{v}{|v|} - \langle \frac{v}{|v|}, x \rangle x.$$

Let δ_v be the dilation of \mathbf{H}_v of coefficient $e^{t \cdot |v|}$. According to Remark 2.2.1, for any $v \in \mathbb{H} \setminus \{0\}$ and $t \in \mathbb{R}$, the family of transformations

$$\Gamma_t^v(x) = ((\Pi_v)^{-1} \circ \delta_v \circ \Pi_v)(x)$$

is a one-parameter family of Möbius transformations of $\mathbb{S}_{\mathbb{H}}$.

Following on the steps of [9], we have

Proposition 3.2.1.

- (i) For t fixed, each Möbius transformation Γ_t^v belongs to $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$.
- (ii) Let Φ_t^v be the flow of $\text{grad}(\varphi_v)$. Then we have $\Phi_t^v = \Gamma_t^v$.
- (iii) For any pair v, w of independent vectors of $\mathbb{H} \setminus \{0\}$, the flow generated by the Lie bracket $[\text{grad}(\varphi_v), \text{grad}(\varphi_w)]$ is a rotation in the plane $P(v, w)$ generated by v and w with rotation angle of value $-t$.

Proof. If I is finite, the proof is given in [9] and [14] so we assume that $I = \mathbb{N}$. Fix some $v \in \mathbb{H}$. We choose a Hilbert basis $\{e_i\}_{i \in \mathbb{N}}$ such that $e_1 = \frac{v}{|v|}$. Then we have $\mathbf{H} \equiv \mathbf{H}_v$ and $\Pi_v \equiv \Pi$. For each n we denote by \mathbf{H}_n the orthogonal of subspace $\{e_i\}$ in \mathbb{H}_n . By induction, we can put on each \mathbf{H}_n an orientation such that the orientation given by \mathbf{H}_n and e_{n+1} is the orientation of \mathbf{H}_{n+1} . Since δ_v preserves the orientation in the restriction to any \mathbf{H}_n , it follows that $(\Pi \circ \Gamma_t^v \circ \Pi^{-1})$ preserves the orientation of \mathbf{H}_n and finally $[\Pi \circ \Gamma_t^v \circ \Pi^{-1}]$ preserves the orientation for any n . Therefore, $A_n = \mathcal{L}^{-1} \circ [\Pi \circ \Gamma_t^v \circ \Pi^{-1}]|_{\mathbf{H}_n}$ belongs to $SO(\mathbb{H}_n, 1)$. Moreover, if $A = \mathcal{L}^{-1} \circ [\Pi \circ \Gamma_t^v \circ \Pi^{-1}]$, then we have $[A]|_{\mathcal{H}_n} = A_n$. This implies that A_n is a Cauchy sequence in G_{∞} for the distance d_{∞} . We deduce that A is the limit of A_n and so A belongs to G_{∞} . This ends the proof of Part (i).

The proof of Part (ii) (resp. Part (iii)) is formally the same as the proof of Lemma 3.1 (resp. Lemma 3.3) of [9] so we will give an abstract of these proofs.

As $\Gamma_t^{\lambda v} = \Gamma_{\lambda t}^v$ without loss of generality we can assume that $|v| = 1$. At first $x = \pm v$ are fixed points for Φ_t^v and Γ_t^v .

Pick some $z \in \mathbb{S}_{\mathbb{H}}$ with $z \neq \pm v$ and let P be the plane in \mathbb{H} generated by v and z . By similar arguments to those in the proof of Lemma 3.1 of [9], we have $\text{grad}\phi_v(x)$ belongs to P for all $x \in P$ and so Φ_t^v preserves P . On the other hand, by construction, Γ_t^v also preserves P . Now, from Lemma 2.2 of [9] we then get that Γ_t^v and Φ_t^v coincide on the circle $P \cap \mathbb{S}_{\mathbb{H}}$, so we get Part (ii).

Let P be the plane generated by v and w , where v, w are independent vectors of $\mathbb{H} \setminus \{0\}$. Since $\Phi_t^v = \Gamma_t^v$ and $\Phi_t^w = \Gamma_t^w$ these flows preserve P , so the Lie bracket $[\text{grad}(\phi_v), \text{grad}(\phi_w)]$ is tangent to P on P . Therefore the flow of $[\text{grad}(\phi_v), \text{grad}(\phi_w)]$ preserves P and according to Lemma 2.2 of [9] in restriction to P , this flow is a rotation with rotation angle of value $-t$. It remains to show that if $x \in \mathbb{S}_{\mathbb{H}}$ is orthogonal to P , this flow keeps x fixed. It reduces to a 3-dimensional problem which can be solved as in the proof of Lemma 3.1 in [9]. \square

Now, If $\{e_i^*\}_{i \in I}$ is the dual basis of $\{e_i\}_{i \in I}$ the map φ_{e_i} is exactly the dual form e_i^* and we denote by ξ_i the gradient of e_i^* . As vector field, we have the decomposition (see [14] and [13]):

$$(3.2.2) \quad \xi_i(z) = \frac{\partial}{\partial x_i} - z_i \sum_{l \in I} z_l \frac{\partial}{\partial x_l}.$$

Therefore the bracket $[\xi_i, \xi_j]$ has the decomposition:

$$(3.2.3) \quad [\xi_i, \xi_j](z) = z_i \frac{\partial}{\partial x_j} - z_j \frac{\partial}{\partial x_i}.$$

Consider the natural action $\mathfrak{A} : \mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}}) \times \mathbb{S}_{\mathbb{H}} \rightarrow \mathbb{S}_{\mathbb{H}}$ on $\mathbb{S}_{\mathbb{H}}$ and we denote by $\mathfrak{a} : \mathfrak{m}_{HS}(\mathbb{S}_{\mathbb{H}}) \rightarrow \text{Vect}(\mathbb{S}_{\mathbb{H}})$ the associated infinitesimal action where $\text{Vect}(\mathbb{S}_{\mathbb{H}})$ is the space of vector fields on $\mathbb{S}_{\mathbb{H}}$. If we identify $\mathfrak{m}_{HS}(\mathbb{S}_{\mathbb{H}})$ with \mathfrak{g}_{∞} , it is classical that we have (cf [10] or [14])

$$\mathfrak{a}([U_i, U_j]) = -[\mathfrak{a}(U_i), \mathfrak{a}(U_j)].$$

As in finite dimension (cf [14]) we have:

Proposition 3.2.2.

- (1) *The action \mathfrak{A} is effective.*⁴
- (2) *The morphism \mathfrak{a} is injective and $\mathfrak{a}(U_i) = \xi_i$.*

Proof.

(1) Let $\phi \in \mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ such that $\phi(z) = z$ for all $z \in \mathbb{S}_{\mathbb{H}}$. According to Proposition 2.1.3, for any n the restriction of ϕ to $\mathbb{H}_n \cap \mathbb{S}_{\mathbb{H}}$ is a Möbius transformation of the finite dimensional sphere $\mathbb{H}_n \cap \mathbb{S}_{\mathbb{H}}$. As in the finite dimensional case this action is effective, the restriction of ϕ to $\mathbb{H}_n \cap \mathbb{S}_{\mathbb{H}}$ is the identity. Therefore the map $\Pi \circ \phi \circ \Pi^{-1}$ from \mathbb{H} to \mathbb{H} is the identity on each subspace $\mathbb{H}_{n-1} = \mathbb{H} \cap \mathbb{H}_n$ for any $n \in \mathbb{N}$. It follows that $\Pi \circ \phi \circ \Pi^{-1} = Id_{\mathbb{H}}$ and then $\phi = Id_{\mathbb{S}_{\mathbb{H}}}$. Therefore the action \mathfrak{A} is effective.

(2) For the injectivity of \mathfrak{a} , see the proof of Proposition 2.9 of [14] part (5). On the other hand according to our identifications, from Proposition 3.2.1, we get $\mathfrak{a}(U_i) = \xi_i$. \square

3.3. On the sub-Riemannian structure on $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$.

Let M be a Hilbert manifold and \mathcal{D} a subbundle of TM . A *sub-Riemannian structure* on M is a triple (M, \mathcal{D}, g) where g is a Riemannian metric on \mathcal{D} . Of course, given a Riemannian metric \bar{g} on M , we get a Riemannian metric g on \mathcal{D} by restriction. On the other hand, there always exists a complementary \mathcal{V} of \mathcal{D} , i.e. $TM = \mathcal{D} \oplus \mathcal{V}$ and so we can extend g into a Riemannian metric \bar{g} on M in an evident way.

Consider any Riemannian metric \bar{g} on M . A curve $\gamma : [0, T] \rightarrow M$ is of class L^1 if we have: $\int_0^T \sqrt{\bar{g}(\dot{\gamma}(t), \dot{\gamma}(t))} dt < \infty$. This property does not depend on the choice of \bar{g} . For such a curve γ , its length $l(\gamma)$ is precisely the quantity $\int_0^T \sqrt{\bar{g}(\dot{\gamma}(t), \dot{\gamma}(t))} dt$ and, of course, $l(\gamma)$ does not depend on its parametrization. A L^1 -curve is called *horizontal* if $\dot{\gamma}(t)$ belongs to $\mathcal{D}(\gamma(t))$. Given any Riemannian metric g on \mathcal{D} , the length of an horizontal curve γ is well defined. Note that we also have

$$(3.3.1) \quad l(\gamma) = \int_0^T |\gamma(t)^{-1} \dot{\gamma}(t)| dt.$$

⁴An action \mathfrak{A} is called effective if $\mathfrak{A}(g, z) = z \quad \forall z$ implies $g = Id$

Given two points x_0 and x_1 in M , let $\mathcal{C}_H(x_0, x_1)$ be the set, eventually empty, of horizontal L^1 -curves $\gamma : [0, T] \rightarrow M$ such that $\gamma(0) = x_0$ and $\gamma(T) = x_1$ for some $T \geq 0$. reparametrization, The *horizontal distance* $d_H(x_0, x_1)$ between x_0 and x_1 is defined by

$$(3.3.2) \quad d_H(x_0, x_1) = \inf \{l(\gamma), \gamma \in \mathcal{C}_H(x_0, x_1)\} \text{ and } d_H(x_0, x_1) = \infty \text{ if } \mathcal{C}_H(x_0, x_1) = \emptyset.$$

In the finite dimension, the infimum in (3.3.2) is always reached. Moreover, the Theorem of Chow gives sufficient conditions under which any two points of M can be joined by a horizontal curve. In this case, d_H becomes a distance.

In infinite dimension, as in the Riemannian case, if $\mathcal{C}_H(x_0, x_1) \neq \emptyset$, the infimum in (3.3.2) could be not reached. Moreover, in this context, to our knowledge, no general result as Chow's theorem exists. Therefore we cannot hope that d_H is a distance in a wide context.

We now come back to the Lie group $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$. The Lie algebra $\mathfrak{m}_{HS}(\mathbb{S}_{\mathbb{H}})$ and $\mathfrak{g} = \mathfrak{so}_{HS}(\mathbb{H}, 1) = \mathfrak{h} \oplus \mathfrak{s}$ being identified and we provide this Lie algebra with the norm $|\cdot|$ associated to the inner product induced by $\frac{1}{2} \langle \cdot, \cdot \rangle$.

Then, the isomorphism $u \rightarrow \begin{pmatrix} 0 & [u]^* \\ [u] & 0 \end{pmatrix}$ from \mathbb{H} to \mathfrak{h} is in fact an isometry. For simplicity, the inner product on \mathfrak{h} will be denoted $\langle \cdot, \cdot \rangle$. Therefore the Hilbert subspace $(\mathfrak{h}, \langle \cdot, \cdot \rangle)$ generates a left invariant distribution Δ and also a left invariant Riemannian metric g on $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ and then $(\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}}), \Delta, g)$ is a sub-Riemannian structure on $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$. Given any $\phi \in \mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$, the *accessibility set* of ϕ is

$$\mathcal{A}(\phi) = \{\psi \in \mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}}) \text{ such that there exists an horizontal curve } \gamma : [0, T] \rightarrow \mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}}) \text{ with } \gamma(0) = \phi \text{ and } \gamma(T) = \psi\}.$$

On the other hand, in the Lie sub-algebra \mathfrak{s} of \mathfrak{g} of $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$, we consider the Banach space

$$\mathfrak{s}_1 = \{P \in \mathfrak{s} \text{ such that } P = \sum_{k,l \in I, k < l} \lambda_{kl} \Omega_{kl}, \sum_{k,l \in I, k < l} |\lambda_{kl}| < \infty\}$$

equipped with the norm $|P|_1 = \sum_{k,l \in I, k < l} |\lambda_{kl}|$. Note that $|P|_1 = \sum_{i \in I} |\langle e_i, P e_i \rangle|$ is the L^1 trace of P and so

$|P|_1$ does not depend on the choice of the fixed Hilbert basis of \mathbb{H} . We denote by $\mathfrak{g}_1 = \mathfrak{h} \oplus \mathfrak{s}_1$ equipped with the norm

$$|(B, P)|_1 = |B| + |P|_1.$$

Of course, the natural inclusion of \mathfrak{g}_1 in \mathfrak{g} is continuous, and the family $\{U_i\}_{i \in I} \cup \{\Omega_{kl}\}_{k,l \in I, k < l}$ is a Schauder basis of \mathfrak{g}_1 . We denote by $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}}) = \text{Exp}(\mathfrak{g}_1)$. Then, it is clear that $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$ has a structure of a Banach Lie group modeled on \mathfrak{g}_1 . Moreover, $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$ is dense in $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$. According to the terminology of weak submanifold of a Banach manifold (cf [13]), we will say that $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$ is a weak Lie subgroup of $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$. Then, we have:

Theorem 3.3.1.

- (i) Any two elements A_0 and A_1 of $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$ can be joined by a horizontal curve.
- (ii) d_H is a distance on $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$.

Proof.

According to the construction of $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$, we can assume that $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}}) = SO_{HS}^1(\mathbb{H}, 1) \subset SO_{HS}(\mathbb{H}, 1)$. On the other hand, it is sufficient to prove part (i) for $A_0 = Id$ and A_1 any point of $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$. Fix some $A \in \mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$. According to Theorem 3.1.2, we have $A = \prod_{j \in J} \text{Exp}(\theta_j B_j) \text{Exp } U$ where each B_j is a finite rank element of \mathfrak{s} and $U \in \mathfrak{h}$.

Therefore the curve $t \rightarrow \text{Exp}(tU)$ is a horizontal curve defined on $[0, 1]$ which joins Id to $\text{Exp}(U)$. It is sufficient to prove the result for $A = \prod_{j \in J} \text{Exp}(\theta_j B_j)$. We denote by $A_j = \text{Exp}(\theta_j B_j)$ and by \mathcal{B} a matrix consisting of blocks

$\bar{B}_j = \theta_j B_j$ in restriction to E_j . Fix such a point A_j . By construction of B_j (cf Appendix 5.1), if we set $\mathbb{E}_j = B_j(\mathbb{H})$, then \mathbb{E}_j is a finite dimensional Hilbert space such that $\ker(B_j) = (\mathbb{E}_j)^\perp$. It follows that $\bar{B}_j = B_j|_{\mathbb{E}_j}$ belongs to $\mathfrak{so}(\mathbb{E}_j, 1)$. Moreover, we also have a decomposition $\mathfrak{so}(\mathbb{E}_j, 1) = \mathfrak{h}_j \oplus \mathfrak{s}_j$ and \bar{B}_j belongs to \mathfrak{s}_j . In the basis of \mathbb{E}_j built in Appendix 5.1, the Lie algebra $\mathfrak{so}(\mathbb{E}_j, 1)$ is generated by $\{\bar{U}_r = U_{rs}|_{\mathbb{E}_j}, l_1 \leq 2l_j\}$ and $\{\bar{\Omega}_{rs} = \Omega_{rs}|_{\mathbb{E}_j}, l_1 \leq r < s \leq l_j\}$. Consider the (left-invariant) sub-Riemannian structure on $SO(\mathbb{E}_j, 1)$ generated by \mathfrak{h}_j , provided with inner product such that $\{\bar{U}_r, r = 1, \dots, n_j\}$ is orthonormal basis. According to the classical Chow theorem, there is a horizontal curve $\bar{\gamma}_j : [0, T_j] \rightarrow SO(\mathbb{E}_j, 1)$ such that $\bar{\gamma}_j(0) = Id_{\mathbb{E}_j}$ and $\bar{\gamma}_j(T_j) = \bar{A}_j = A_j|_{\mathbb{E}_j}$. Consider $\gamma_j : [0, T_j] \rightarrow SO_{HS}(\mathbb{H}, 1)$ defined by $\gamma_j(t)|_{\mathbb{E}_j} = \bar{\gamma}_j(t)$ and $\gamma_j(t)|_{(\mathbb{E}_j)^\perp} = Id_{(\mathbb{E}_j)^\perp}$.

Then γ_j is a horizontal curve which joins $Id_{\mathbb{H}}$ to A_j .

If J is finite, we can assume that $J = \{1, \dots, N\}$ otherwise, we can assume that $J = \mathbb{N}$. We parameterize γ_j into a curve c_j on $[\tau_{j-1}, \tau_j]$ by setting $c_j(s) = \gamma_j(s - \tau_{j-1})$.

For each integer $n \in J$, consider the finite composition $C_n : [0, \tau_n] \rightarrow SO_{HS}(\mathbb{H}, 1)$ inductively defined by $C_n(s) = C_{n-1}(s)$ for $s \in [0, \tau_{n-1}]$,

$$C_n(s) = c_n(s)C_{n-1}(\tau_{n-1}) \text{ for } s \in [\tau_{n-1}, \tau_n].$$

Then C_n is a L^1 horizontal curve which joins $Id_{\mathbb{H}}$ to $\prod_{j=1}^n A_j$. Therefore if J is finite the proof is complete.

Assume now that $J = \mathbb{N}$. We set $\tau = \lim_{n \rightarrow \infty} \tau_n$ if this limit is finite otherwise we set $\tau = \infty$. We must show that

$\lim_{n \rightarrow \infty} C_n(s)$ is well defined for all $s \in [0, \tau]$. At first, as $A = \lim_{n \rightarrow \infty} \prod_{j=1}^n A_j$ we have then $\lim_{n \rightarrow \infty} C_n(\tau_n) = A$.

But, by construction, for each $m > n$, we have $C_m|_{[0, \tau_n]} = C_n$. So, for any $s \in [0, \tau[$ there exists n such that $s \in [0, \tau_n]$ and so $C(s) = C_n(s)$ is well defined. Of course, such a construction is differentiable almost every where but without a good choice for each curve $\bar{\gamma}_j$, in general, $\tau = \infty$ and even if τ is finite, C is not of class L^1 and in particular, we can have

$$\lim_{n \rightarrow \infty} \int_0^{\tau_n} \sqrt{g(\dot{C}_n(s), \dot{C}_n(s))} ds = \infty.$$

To end this proof, we will use the results about the sub-Riemannian structure of $SU(1, 1)$ to get the following Lemma.

Lemma 3.3.1. *For each j , with the previous notations, we can choose an horizontal curve $\bar{\gamma}_j: [0, T_j] \rightarrow SO(\mathbb{E}_j, 1)$ arc-length parametrized such that*

$$\bar{\gamma}_j(0) = Id_{\mathbb{E}_j}, \quad \bar{\gamma}_j(T_j) = \bar{A}_j, \quad \text{and } l(\bar{\gamma}_j) = n_j |\theta_j| = T_j.$$

For each $j \in J$, $C_n(\tau_n) = \prod_{j=1}^n A_j$. Therefore, by construction of the family A_j , the endomorphism $C_n(\tau_n)$ is an isometry of \mathbb{H} which preserves the space $\mathbb{K} \oplus \mathbb{E}_1 \oplus \dots \oplus \mathbb{E}_n$. Since γ_{n+1} is arc-length parametrized we have:

$$(3.3.3) \quad \int_{\tau_n}^{\tau_{n+1}} \sqrt{g(\dot{C}(s), \dot{C}(s))} ds = \int_0^{T_{n+1}} \sqrt{g(\dot{\gamma}_{n+1}(s), \dot{\gamma}_{n+1}(s))} ds = T_j = n_j |\theta_j|$$

But from the decomposition of \mathcal{B} ,

$$|\mathcal{B}|_1 = 2 \sum_{j \in J} n_j |\theta_j|.$$

and, according to (3.3.3) and the construction of C we have then

$$l(C) = \int_0^\tau \sqrt{g(\dot{C}(s), \dot{C}(s))} ds = \sum_{j \in J} \int_{\tau_{j1}}^{\tau_j} \sqrt{g(\dot{C}(s), \dot{C}(s))} ds = \frac{1}{2} |\mathcal{B}|_1.$$

This ends the proof of Part (i).

From the definition of the distance d on $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$, for any ψ and ψ' in $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$ we have $d_H(\psi, \psi') \geq d(\psi, \psi')$. As from part (i) the restriction of d_H to $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$ it follows easily that d_H is a distance and then Part (ii) is proved. \square

4. CONTROL PROBLEM OF A HILBERT SNAKE AND ACCESSIBILITY SETS

4.1. The configuration space.

Again in this section, the Hilbert basis $\{e_i\}_{i \in \mathbb{N}}$ in \mathbb{H} is fixed.

A curve $\gamma: [a, b] \rightarrow \mathbb{H}$ (not necessary continuous) is called C^k -piecewise if there exists a finite set $\mathcal{P} = \{a = s_0 < s_1 < \dots < s_N = b\}$ such that, for all $i = 0, \dots, N-1$, the restriction of γ to the interval $]s_i, s_{i+1}[$ can be extended to a curve of class C^k on the closed interval $[s_i, s_{i+1}]$. Given any metric space (X, d) and partition $\mathcal{P} = \{a = s_0 < s_1 < \dots < s_N = b\}$ of $[a, b]$, let $\mathcal{C}_{\mathcal{P}}^k([a, b], X)$ be the set of curves $u: [0, L] \rightarrow X$ which are C^k -piecewise relatively to \mathcal{P} for $k \in \mathbb{N}$ and equipped with the distance

$$\delta(u_1, u_2) = \sup_{t \in [0, L]} d(u_1(t), u_2(t)).$$

Note that, if $\mathcal{P} = \{0, L\}$ and if X is a submanifold of \mathbb{H} , then $\mathcal{C}_{\mathcal{P}}^k([a, b], X)$ is the space of continuous $C^k([0, L], X)$ curves from $[0, L]$ to X of class C^k , and as in finite dimension, we have a natural structure of Banach manifold on $C^k([0, L], X)$.

Throughout this paper, we fix a real number $L > 0$ and \mathcal{P} is a given fixed partition of $[0, L]$.

A Hilbert snake is a continuous piecewise C^1 -curve $S: [0, L] \rightarrow \mathbb{H}$, such that $\|\dot{S}(t)\| = 1$ and $S(0) = 0$. In fact, a snake is characterized by $u(t) = \dot{S}(t)$ and of course we have $S(t) = \int_0^t u(s) ds$ where $u: [0, L] \rightarrow \mathbb{S}^\infty$ is a

piecewise C^0 -curve associated to the partition \mathcal{P} . Moreover, this snake is affine if and only if u is constant on each subinterval of \mathcal{P} . The set $\mathcal{C}_{\mathcal{P}}^L = \mathcal{C}_{\mathcal{P}}^0([0, L], \mathbb{S}_{\mathbb{H}})$ is called the **configuration space of the snake** in \mathbb{H} of length L relative to the partition \mathcal{P} .

The map $u \mapsto (u|_{[s_0, s_1]}, \dots, u|_{[s_i, s_{i+1}]}, u|_{[s_{N-1}, s_N]})$ is an homeomorphism between $\mathcal{C}_{\mathcal{P}}^L$ and $\prod_{i=0}^{N-1} C^0([s_i, s_{i+1}], \mathbb{S}^{\infty})$. Moreover, this map permits to put on $\mathcal{C}_{\mathcal{P}}^L$ a structure of Banach manifold diffeomorphic to the Banach product structure $\prod_{i=0}^{N-1} C^0([s_i, s_{i+1}], \mathbb{S}^{\infty})$. The tangent space $T_u \mathcal{C}_{\mathcal{P}}^L$ can be identified with the set

$$\{v \in \mathcal{C}_{\mathcal{P}}^0([0, L], \mathbb{H}) \text{ such that } \langle u(s), v(s) \rangle = 0 \text{ for all } s \in [0, L]\}.$$

This space is naturally provided with two non equivalent norms

the natural $\|\cdot\|_{\infty}$

the $\|\cdot\|_{L^2}$ associated to the inner product $\langle v, w \rangle_{L^2} = \int_0^L \langle v(s), w(s) \rangle ds$.

4.2. The horizontal distribution associated to a Hilbert snake.

For any $u \in \mathcal{C}_{\mathcal{P}}^L$ the **Hilbert snake** associated to u is the map $S_u : [0, L] \rightarrow \mathbb{H}$ defined by

$$S_u(t) = \int_0^t u(s) ds. \text{ The } \mathbf{endpoint \ map: } \mathcal{E} : \mathcal{C}_{\mathcal{P}}^L \rightarrow \mathbb{H} \text{ defined by } u \rightarrow S_u(L)$$

is smooth and we have $T_u \mathcal{E}(v) = \int_0^L v(s) ds$.

Let \mathcal{D}_u be the orthogonal of $\ker T_u \mathcal{E}$ (for the inner product $\langle \cdot, \cdot \rangle_{L^2}$ on $T_u \mathcal{C}_{\mathcal{P}}^L$). Then we have the decomposition

$$T_u \mathcal{C}_{\mathcal{P}}^L = \mathcal{D}_u \oplus \ker T_u \mathcal{E}$$

and the restriction of $T_u \mathcal{E}$ to \mathcal{D}_u is a continuous injective morphism into \mathbb{H} . The family $u \mapsto \mathcal{D}_u$ is a (closed) distribution on $\mathcal{C}_{\mathcal{P}}^L$ called the **horizontal distribution**, and each vector field X (resp. curve) on $\mathcal{C}_{\mathcal{P}}^L$ which is tangent to \mathcal{D} is called a **horizontal vector field** (resp. **horizontal curve**).

The inner product on \mathbb{H} gives rise to a Riemannian metric g on $T\mathbb{H} \equiv \mathbb{H} \times \mathbb{H}$ given by $g_x(u, v) = \langle u, v \rangle$. Let $\phi : \mathbb{H} \rightarrow \mathbb{R}$ be a smooth function. The usual gradient of ϕ on \mathbb{H} is the vector field

$$\text{grad}(\phi) = (g^b)^{-1}(d\phi),$$

where g^b is the canonical isomorphism of bundle from $T\mathbb{H}$ to its dual bundle $T^*\mathbb{H}$, corresponding to the Riesz representation i.e. $g^b(v)(w) = \langle v, w \rangle$. Thus $\text{grad}(\phi)$ is characterized by:

$$(4.2.1) \quad g(\text{grad}(\phi), v) = \langle \text{grad}(\phi), v \rangle = d\phi(v),$$

for any $v \in \mathbb{H}$.

In the same way, to the inner product on $T\mathcal{C}_{\mathcal{P}}^L$ (previously defined), is associated a *weak* Riemannian metric G and we cannot define in the same way the gradient of any smooth function on $\mathcal{C}_{\mathcal{P}}^L$. However, let $G^b : T\mathcal{C}_{\mathcal{P}}^L \rightarrow T^*\mathcal{C}_{\mathcal{P}}^L$ be the morphism bundle defined by:

$$G_u^b(v)(w) = G_u(v, w)$$

for any v and w in $T_u \mathcal{C}_{\mathcal{P}}^L$. Given any smooth function $\phi : \mathbb{H} \rightarrow \mathbb{R}$, then $\ker d(\phi \circ \mathcal{E})$ contains $\ker T\mathcal{E}$ and so belongs to $G_u^b(T_u \mathcal{C}_{\mathcal{P}}^L)$. Moreover,

$$(4.2.2) \quad \nabla \phi = (G^b)^{-1}(d(\phi \circ \mathcal{E}))$$

is tangent to \mathcal{D}_u , and we have

$$(4.2.3) \quad \nabla \phi(u)(s) = \text{grad}(\phi)(\mathcal{E}(u)) - \langle \text{grad}(\phi)(\mathcal{E}(u)), u(s) \rangle u(s).$$

The vector field $\nabla \phi$ is called **horizontal gradient** of ϕ .

To each vector $x \in \mathbb{H}$, we can associate the linear form x^* such that $x^*(z) = \langle z, x \rangle$. This implies that the horizontal gradient ∇x^* is well defined. In particular,

Observation 4.2.1.

To each vector e_i , $i \in \mathbb{N}$, of the Hilbert basis, we can associate the horizontal vector field $E_i = \nabla e_i^*$. In fact, the family $\{E_i\}_{i \in \mathbb{N}}$ of vector fields generates the distribution \mathcal{D} .

4.3. Set of critical values and set of singular points of the endpoint map.

As the continuous linear map $T_u \mathcal{E} : T_u \mathcal{C}_P^L \rightarrow T_{\mathcal{E}(u)} \mathbb{H} \equiv \mathbb{H}$ is closed it follows that $\rho_u = T_u \mathcal{E}|_{\mathcal{D}_u}$ is an isomorphism from \mathcal{D}_u to the closed subset $\rho_u(\mathcal{D}_u)$ of \mathbb{H} .

Consider the decompositions $z = \sum_{i \in \mathbb{N}} z_i e_i$ and $u(s) = \sum_{i \in \mathbb{N}} u_i(s) e_i$. Then u is singular if and only if (cf [13]):

$$(4.3.1) \quad Lz_i = \sum_{j \in \mathbb{N}} \int_0^L u_i(s) u_j(s) z_j ds \quad \forall i \in \mathbb{N}.$$

Let Γ_u be the endomorphism defined by matrix of general term $(\int_0^L u_i(s) u_j(s) ds)$. Note that Γ_u is self-adjoint. The endomorphism $A_u = L.Id - \Gamma_u$ is also self-adjoint and, in fact, its matrix in the basis $\{e_i\}_{i \in \mathbb{N}}$ is $(L\delta_{ij} - \int_0^L u_i(s) u_j(s) ds)$. It follows that (4.3.1) is equivalent to

$$(4.3.2) \quad A_u(z) = 0.$$

Finally, u is a singular point if and only if L is an eigenvalue of Γ_u and also if and only if the vector space generated by $u([0, L])$ is 1-dimensional.

The image of \mathcal{E} is the closed ball $B(0, L)$ in \mathbb{H} and **set of critical values** of \mathcal{E} is the union of spheres $S(0, L_j)$ for $j = 1, \dots, n$ with $0 \leq L_j \leq L$.

Finally we obtain the following result ([13]):

Proposition 4.3.1.

- (1) *The set $\mathcal{R}(\mathcal{E})$ (resp. $\mathcal{V}(\mathcal{E})$) of regular values (resp. points) of \mathcal{E} is an open dense subset of \mathcal{C}_P^L (resp. \mathbb{H}).*
- (2) *For any $u \in \mathcal{R}(\mathcal{E})$ the linear map $\rho_u : \mathcal{D}_u \rightarrow \{\mathcal{E}(u)\} \times \mathbb{H}$ is an isomorphism, and on \mathcal{D}_u , the inner product induced by $\langle \cdot, \cdot \rangle_{L^2}$ and the inner product defined ρ_u from \mathbb{H} are equivalent. Moreover the distribution $\mathcal{D}|_{\mathcal{R}(\mathcal{E})}$ is a trivial Hilbert bundle over $\mathcal{R}(\mathcal{E})$ which is isometrically isomorphic to $T\mathbb{P}^\infty$.*

4.4. Accessibility results for a Hilbert snake.

Recall that given any continuous piecewise C^k -curve $c : [0, T] \rightarrow \mathbb{H}$, a lift of c is a continuous piecewise C^k -curve $\gamma : [0, T] \rightarrow \mathcal{C}_P^L$ such that $\mathcal{E}(\gamma(t)) = c(t)$. Thus, for a Hilbert snake we can consider the following optimal control problem :

Given any continuous piecewise C^k -curve $c : [0, T] \rightarrow \mathbb{H}$, we look for a lift $\gamma : [0, 1] \rightarrow \mathcal{C}_P^L$, say $t \rightarrow u_t$, such that, for all $t \in [0, 1]$,

- the associated family $S_t = \int_0^L u_t(s) ds$ of snakes satisfies $S_t(L) = c(t)$ for all $t \in [0, 1]$,
- the **infinitesimal kinematic energy**: $\frac{1}{2} \|\dot{\gamma}(t)\|_{L^2}^2 = \frac{1}{2} G(\dot{\gamma}(t), \dot{\gamma}(t))$ is minimal.

Then such a type of optimal problem has a solution if and only if the curve c has a horizontal lift. We shall say that such a horizontal lift is an **optimal control**.

On the other hand, we can also ask when two positions x_0 and x_1 of the "head" of the snake can be joined by a continuous piecewise smooth curve c which has an optimal control γ as lift. As in finite dimension, the **accessibility set** $\mathcal{A}(u)$, for some $u \in \mathcal{C}_P^L$, is the set of endpoints $\gamma(T)$ for any piecewise smooth horizontal curve $\gamma : [0, T] \rightarrow \mathcal{C}_P^L$ such that $\gamma(0) = u$. In this case if $x_0 = S_u(L)$ then any $z = S_{u'}(L)$ can be joined from x_0 by an absolutely continuous curve c which has an optimal control when u' belongs to $\mathcal{A}(u)$.

When \mathbb{H} is finite dimensional, the set $\mathcal{A}(u)$ is exactly the orbit of the action. In finite dimension, given any horizontal distribution \mathcal{D} on a finite dimensional manifold M , the famous *Sussmann's Theorem* (see [16]) asserts that each accessibility set is a smooth immersed manifold which is an integral manifold of a distribution $\hat{\mathcal{D}}$ which contains \mathcal{D} (i.e. $\mathcal{D}_x \subset \hat{\mathcal{D}}_x$ for any $x \in M$) and characterized by:

$\hat{\mathcal{D}}$ is the smallest distribution which contains \mathcal{D} and which is invariant by the flow of any (local) vector field tangent to \mathcal{D} .

From this argument, E. Rodriguez proved that the set $\mathcal{A}(u)$ is an immersed finite dimensional submanifold of \mathcal{C}_P^L in [14].

In the context of Banach manifolds the reader can find some generalization of this Sussmann's result in [12]. Unfortunately, in our context, this last results give only some density results on accessibility sets, with analogue construction as in finite dimension case (see [13]).

Precisely, according to observation 4.2.1, to each Hilbert basis $\{e_i, i \in \mathbb{N}\}$ the family $\mathcal{X} = \{E_i, i \in \mathbb{N}\}$ of (global) vector fields on \mathcal{C}_P^L generates the horizontal distribution \mathcal{D} . On the other hand the family

$$\mathcal{Y} = \mathcal{X} \bigcup \{[E_i, E_j], i, j \in I, i < j\}$$

generates a weak Hilbert distribution $\bar{\mathcal{D}}$ on \mathcal{C}_P^L . Then we have:

Theorem 4.4.1. [13]

The distribution $\bar{\mathcal{D}}$ has the following properties:

- (i) $\bar{\mathcal{D}}$ does not depend on the choice of the basis $\{e_i, i \in \mathbb{N}\}$;
- (ii) $\bar{\mathcal{D}}_x$ is dense in $\bar{\mathcal{D}}_x$ for all $x \in M$;
- (iii) $\bar{\mathcal{D}}$ is integrable;
- (iv) the accessibility set $\mathcal{A}(u)$ of a point u of any maximal integral manifold N of $\bar{\mathcal{D}}$ is a dense subset of N .

In the following section we will give a new proof of this Theorem which use the natural action of $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ on \mathcal{C}_P^L , the sub-Riemannian structure of $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ and Theorem 3.3.1. We also get a geometrical interpretation of the maximal integral manifold of \mathcal{D}

4.5. Action of $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ on \mathcal{C}_P^L and proof of Theorem 1.

Since a configuration $u \in \mathcal{C}_P^L$ is a curve $u : [0, L] \rightarrow \mathbb{S}_{\mathbb{H}}$, we can naturally define an action of $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ on \mathcal{C}_P^L (again denote by \mathfrak{A}) by

$$\mathfrak{A}(\phi, u)(s) = \phi(u(s)) \text{ for } s \in [0, L].$$

Since the action of $\mathfrak{M}_{HS}(\mathbb{S}_{\mathbb{H}})$ on $\mathbb{S}_{\mathbb{H}}$ is smooth and effective, the same is true for the action on \mathcal{C}_P^L .

Let $\mathfrak{a} : \mathfrak{m}_{HS}(\mathbb{S}_{\mathbb{H}}) \rightarrow \text{Vect}(\mathcal{C}_P^L)$ be the associated infinitesimal action where $\text{Vect}(\mathcal{C}_P^L)$ denote the space of vector fields on \mathcal{C}_P^L . As previously, we identify $\mathfrak{m}_{HS}(\mathbb{S}_{\mathbb{H}})$ with \mathfrak{g}_{∞} , and we have (cf [9] or [14])

$$\mathfrak{a}([U_i, U_j]) = -[\mathfrak{a}(U_i), \mathfrak{a}(U_j)].$$

Moreover, according to Proposition 3.2.2 and the characterization (4.2.3) of $\text{grad}\phi$ and the definition of E_i , we have

$$(4.5.1) \quad \mathfrak{a}(U_i) = E_i \text{ and } \mathfrak{a}([U_i, U_j]) = \mathfrak{a}(\Omega_{ij}) = -[E_i, E_j].$$

Of course, we also have a bundle morphism (again denoted \mathfrak{a}):

$$\mathfrak{a} : \mathfrak{g} \times \mathcal{C}_P^L \rightarrow T\mathcal{C}_P^L.$$

Now, we consider the restriction \mathfrak{A}^1 of the previous action \mathfrak{A} to $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$ on \mathcal{C}_P^L and we also have the same relation (4.5.1) for the restriction \mathfrak{a}^1 of \mathfrak{a} to the Lie algebra $\mathfrak{g}_1 = \mathfrak{m}_{HS}^1(\mathbb{S}_{\mathbb{H}})$ of $\mathfrak{M}_{HS}^1(\mathbb{S}_{\mathbb{H}})$.

According to the notations of Section 4.3 of [13] the Banach space \mathbb{G}^2 is isomorphic to \mathfrak{g} . Therefore we have $\mathfrak{a}(\mathfrak{g} \times \{u\}) = \mathcal{D}(u)$. Therefore, from the proof of Lemma 4.4 and Claim 1, we obtain that the orbit of the action \mathfrak{A} through u is exactly the maximal integral manifold of \mathcal{D} through $u \in \mathcal{C}_P^L$.

On the other hand the orbit $\mathcal{O}^1(u)$ of the action \mathfrak{A}^1 through $u \in \mathcal{C}_P^L$ is contained in the orbit $\mathcal{O}(u)$ of \mathfrak{A} through u . Moreover $\mathcal{O}^1(u)$ is dense in $\mathcal{O}(u)$. But according to Theorem 3.3.1 we can obtain the inclusion $\mathcal{O}^1(u) \subset \mathcal{A}(u)$. Therefore the proof of Theorem 1 is complete.

5. APPENDIX

5.1. Appendix A1: proof of Theorem 3.1.2.

Given a Hilbert space \mathbb{H} , we denote by $SO_{HS}(\mathbb{H})$ the Hilbert-Schmidt Lie group $SO(\mathbb{H}) \cap GL_{HS}(\mathbb{H})$ provided with the topology of the Hilbert-Schmidt norm and by $\mathfrak{so}_{HS}(\mathbb{H})$ its Lie algebra. At first we prove the following result (cf [6] for finite dimension)

Proposition 5.1.1.

The map $\text{Exp} : \mathfrak{so}_{HS}(\mathbb{H}) \rightarrow SO_{HS}(\mathbb{H})$ is surjective. More, precisely for each $Q \in SO_{HS}(\mathbb{H})$, there exists a family $\{\theta_j\}_{j \in J}$ with $0 < \theta_j \leq \pi$ and a family of $\{B_j\}_{j \in J}$ with $B_j \in \mathfrak{so}_{HS}(\mathbb{H})$ such that $[B_k, B_j] = 0$ for $k \neq j$ and $(B_j)^3 = -B_j$ so that

$$Q = \prod_{j \in J} \text{Exp}(\theta_j B_j) = \text{Exp}\left(\sum_{j \in J} \theta_j B_j\right).$$

Moreover, if n_j is the rank of B_j then $(|B|_{HS})^2 = \sum_{j \in J} n_j (\theta_j)^2$, where $B = (\sum_{j \in J} \theta_j B_j)$.

Proof. Let $B \in \mathfrak{so}_{HS}(\mathbb{H})$, B is a compact operator skew-adjoint. Therefore, in the complexification \mathbb{H}^C of \mathbb{H} , we can write $B = iA$, where A is a self adjoint compact operator. It follows that the eigenvalues of B are of type $\{\pm i\lambda_j\}_{j \in J}$ where J is a finite or countable set and $\{\lambda_j\}$ is a strictly positive decreasing sequence which converges to 0 if J is countable. From classical spectral theory we have:

$$(5.1.1) \quad \mathbb{H} = \bigoplus_{j \in J} \mathbb{E}_j \oplus \mathbb{K}$$

where \mathbb{E}_j is the subspace such that the restriction of B to \mathbb{E}_j is $\pm i\lambda_j Id_{\mathbb{E}_j}$ and \mathbb{K} is the kernel of B . Moreover, each \mathbb{E}_j is orthogonal to \mathbb{E}_k and \mathbb{K} for $k \neq j$. In particular, \mathbb{E}_j is an even finite dimensional space. We can choose a Hilbert basis $\cup_{j \in J} \{e_{l_1}, \dots, e_{2l_j}\} \cup \{e_l, l \in L\}$ of \mathbb{H} such that $\{e_{l_1}, \dots, e_{2l_j}\}$ is a basis of \mathbb{E}_j and $\{e_l, l \in L\}$ is a basis of \mathbb{K} . Moreover such a choice can be done such that the restriction of B to \mathbb{E}_j is of type $\lambda_j \bar{B}_j$ where \bar{B}_j has a matrix of the form

$$(5.1.2) \quad \begin{pmatrix} J_{l_1} & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & J_{l_r} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & J_{l_j} \end{pmatrix}$$

where each block $J_{l_r} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From this construction, we see that $\{\pm i\lambda_j\}_{j \in J}$ is the set of non zero eigenvalues of B and \mathbb{E}_j is the eigenspace associated to $\pm i\lambda_j$.

Let B_j be the endomorphism whose restriction to \mathbb{E}_j is $\frac{1}{\lambda_j} B|_{\mathbb{E}_j}$ and which is 0 on $(\mathbb{E}_j)^\perp$. By construction, we have:

$$B = \sum_{j \in J} \lambda_j B_j, \quad [B_k, B_j] = 0, \text{ for } k \neq j, \text{ and } (B_j)^3 = -B_j.$$

It follows that we get

$$(5.1.3) \quad Q = \exp B = \exp\left(\sum_{j \in J} \lambda_j B_j\right) = \prod_{j \in J} \exp(\lambda_j B_j).$$

In particular, the eigenvalues of Q which are different from 1 is the family $e^{\pm i\lambda_j}$. Thus in (5.1.3) each $e^{\pm i\lambda_j}$ can be written $e^{\pm i\theta_j}$ with $0 < \theta_j \leq \pi$. and we have

$$(|B|_{HS})^2 = 2 \sum_{j \in J} n_j (\theta_j)^2$$

where $n_j = \dim \mathbb{E}_j$.

Conversely, consider any $Q \in SO_{HS}(\mathbb{H})$. Then, $C = Q - Id$ is compact and so the set of eigenvalues of Q different from 1 is at most countable. Since Q is unitary of a real Hilbert space, we can write this set as $\{e^{\pm i\theta_j}\}_{j \in J}$. Note that each eigenspace of Q is an eigenspace of C and conversely. Moreover, the set of non zero eigenvalues of C is $\{e^{\pm i\theta_j} - 1\}_{j \in J}$. Therefore we have a spectral decomposition associated to C of type (5.1.1) where \mathbb{K} is the kernel of C . Note that the restriction Q_j of Q to each finite dimensional space \mathbb{E}_j is an isometry of this space whose eigenvalues are $\{e^{\pm i\theta_j}\}$. According to the classical Lemma of decomposition of rotations in finite dimension, (see [1] for instance), we have an orthogonal basis $\{e_{l_1}, \dots, e_{2l_j}\}$ of \mathbb{E}_j in which Q_j has a matrix of the form:

$$\begin{pmatrix} R_{l_1} & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & R_{l_r} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & R_{l_j} \end{pmatrix}$$

where each block $R_{l_r} = \begin{pmatrix} \cos \theta_{l_r} & -\sin \theta_{l_r} \\ \sin \theta_{l_r} & \cos \theta_{l_r} \end{pmatrix}$. In fact we must have

$$\theta_{l_r} \equiv \theta_j \pmod{\pi}.$$

It follows that we have $Q_j = \exp(\theta_j \bar{B}_j)$ where \bar{B}_j has a matrix of type (5.1.2) in the previous basis. As in the first part, let B_j be the endomorphism which is equal to \bar{B}_j on \mathbb{E}_j and is zero on $(\mathbb{E}_j)^\perp$.

On the other hand, let \hat{Q}_j be the invertible operator whose restriction to \mathbb{E}_j is equal to Q_j and which is the identity on $(\mathbb{E}_j)^\perp$. Of course the infinite composition $\prod_{j \in J} \hat{Q}_j$ is equal to Q and we get

$$Q = \prod_{j \in J} \exp(\theta_j B_j).$$

As in the first part, by construction, we again have $[B_k, B_j] = 0$ it follows that $B = \sum_{j \in J} \theta_j B_j$ is well defined and $|B|_{HS}^2 = 2 \sum_{j \in J} n_j (\theta_j)^2$.

□

We also need the following result (see [5] for finite dimension).

Proposition 5.1.2.

Given a boost $T \in SO_{HS}(\mathbb{H}, 1)$, there exists $U \in \mathfrak{h}$ such that $T = \text{Exp}(U)$.

The proof of this Proposition is a formal adaptation of the corresponding result in finite dimension of [5]. We only give the essential arguments.

Proof. Let $U \in \mathfrak{h}$. We have $U = \begin{pmatrix} 0 & [u]^* \\ [u] & 0 \end{pmatrix}$ where $u \in \mathbb{H}$. We have $U^3 = \omega^2 U$ where $\omega = |u|$. By application of this relation we easily get

$$\text{Exp}(U) = Id_{\mathcal{H}} + \frac{\sinh \omega}{\omega} U + \frac{\cosh \omega - 1}{\omega^2} U^2.$$

As in finite dimension we obtain:

$$\text{Exp}(U) = \begin{pmatrix} \cosh \omega & \frac{\sinh \omega}{\omega} [u]^* \\ \frac{\sinh \omega}{\omega} [u] & Id_{\mathbb{H}} + \frac{\cosh \omega - 1}{\omega^2} [u][u]^* \end{pmatrix}.$$

We have the relation

$$\left(Id_{\mathbb{H}} + \frac{\cosh \omega - 1}{\omega^2} [u][u]^* \right)^2 = Id_{\mathbb{H}} + \frac{\sinh^2 \omega}{\omega^2} [u][u]^*$$

Finally, we get

$$\text{Exp}(U) = \begin{pmatrix} \cosh \omega & \frac{\sinh \omega}{\omega} [u]^* \\ \frac{\sinh \omega}{\omega} [u] & \sqrt{Id_{\mathbb{H}} + \frac{\sinh^2 \omega}{\omega^2} [u][u]^*} \end{pmatrix}.$$

On the other hand, from the proof of Proposition 2.2.1 we have $T = \begin{pmatrix} c & [v]^* \\ [v] & \sqrt{Id_{\mathbb{H}} + [v].[v]^*} \end{pmatrix}$ for some $v \in \mathbb{H}$. Given $v \in \mathbb{H}$ we have then to find $u \in \mathbb{H}$ which satisfies the following equation:

$$\begin{pmatrix} c & [v]^* \\ [v] & \sqrt{Id_{\mathbb{H}} + [v].[v]^*} \end{pmatrix} = \begin{pmatrix} \cosh \omega & \frac{\sinh \omega}{\omega} [u]^* \\ \frac{\sinh \omega}{\omega} [u] & \sqrt{Id_{\mathbb{H}} + \frac{\sinh^2 \omega}{\omega^2} [u][u]^*} \end{pmatrix}.$$

This equation can be solved as in finite dimension, point by point (cf [5]).

□

Proof of Theorem 3.1.2.

Let $A \in SO_{HS}(\mathbb{H}, 1)$. From Proposition 3.1.1, we have $A = PT$ where $P = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$ and Q belongs to $SO(\mathbb{H})$ and where T is a boost.

From Proposition 5.1.2, there exists a family of endomorphisms $\{B_j\}_{j \in J}$ with $B_j \in \mathfrak{so}_{HS}(\mathbb{H})$ such that $[B_k, B_j] = 0$ for $j \neq k$ and a sequence $\{\theta_j\}_{j \in J}$ with $0 < \theta_j \leq \pi$ so that $Q = \prod_{j \in J} \text{Exp}(\theta_j B_j)$. According to the

isomorphism $Q \rightarrow P = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$ from $\mathfrak{so}_{HS}(\mathbb{H})$ to \mathfrak{s} , we may assume that B_j belongs to \mathfrak{s} . It follows that we get

$$P = \prod_{j \in J} \text{Exp}(\theta_j B_j).$$

According to Proposition 5.1.2 the proof is complete.

□

5.2. Appendix A2: proof of Lemma 3.3.1.

We first recall some result about sub-Riemannian geometry on $SU(1,1)$. At first, we can identify \mathbb{R}^2 with the complex space \mathbb{C} it is classical that $SO_0(2,1)$ is isomorphic to $PSU(1,1)$ which is the connected components of the identity of the Lie group $SU(1,1)$. It follows that $SU(1,1)$ is the group of invertible matrices of type $\begin{pmatrix} z_1 & z_2 \\ \bar{z}_1 & \bar{z}_2 \end{pmatrix}$ where z_1 and z_2 belongs to \mathbb{C} . Note that $SU(1,1)$ can be identified with $\mathbb{C} \times \mathbb{S}^1$. The Lie algebra $\mathfrak{su}(1,1)$ of $SU(1,1)$ is generated by:

$$X = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad Y = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad Z = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

We have the bracket relations

$$[X, Y] = -Z, \quad [X, Z] = -Y, \quad [Y, Z] = X,$$

On $SU(1,1)$ we consider the left invariant distribution Δ generated by X and Y and the left invariant Riemannian metric induced by $\frac{1}{2}Tr(X_1X_2)$ on the subspace generated by X and Y . We get a sub-Riemannian structure $(SU(1,1), \Delta, g)$ on $SU(1,1)$. Let δ be the left-invariant horizontal distance associate to this structure. The universal covering $\tilde{S}U(1,1)$ can be identified with $\mathbb{C} \times \mathbb{R}$. The canonical projection $\rho : \tilde{S}U(1,1) \rightarrow SU(1,1)$ is given by:

$$(z, t) \rightarrow \begin{pmatrix} \sqrt{1+|z|^2}e^{it} & z \\ \bar{z} & \sqrt{1+|z|^2}e^{-it} \end{pmatrix}$$

For our purpose, we need only the following partial result of [8]:

Proposition 5.2.1.

Let $A = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$ with $t \neq 0$. There exists a (normal) minimal length horizontal geodesic which joins Id to A and the horizontal distance $\delta(Id, A) = |\theta|$ for $0 < |\theta| \leq \pi$ and $\pm\theta \equiv t \pmod{\pi}$.

Now, we have an isomorphism from $SU(1,1)$ to $SO(2,1)$ given by:

$$\begin{pmatrix} \sqrt{1+|z|^2}e^{it} & z \\ \bar{z} & \sqrt{1+|z|^2}e^{-it} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-it} \end{pmatrix} \begin{pmatrix} \sqrt{1+|z|^2} & \text{Re}(z) & \text{Im}(z) \\ \text{Re}(z) & a & b \\ \text{Im}(z) & b & c \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ b & c \end{pmatrix}^2 = \begin{pmatrix} \text{Re}(z)^2 & \text{Re}(z)\text{Im}(z) \\ \text{Re}(z)\text{Im}(z) & \text{Im}(z)^2 \end{pmatrix}$.

The induced isomorphism between Lie algebra is then $\begin{pmatrix} it & z \\ \bar{z} & -it \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \text{Re}(z) & \text{Im}(z) \\ \text{Re}(z) & \cos t & -\sin t \\ \text{Im}(z) & \sin t & \cos t \end{pmatrix}$. As a consequence we get an isomorphism between the sub-Riemannian structure on $SU(1,1)$ and the sub-Riemannian structure on $SO(2,1)$.

Proof of Lemma 3.3.1.

Recall that $\bar{A}_j = \begin{pmatrix} 1 & 0 \\ 0 & \text{Exp}(\bar{B}_j) \end{pmatrix}$ and \bar{B}_j has a decomposition (5.1.2) in diagonal blocks $\theta_j J_{l_r}$. Each block J_{l_r} gives rise to an element of $SO(\mathbb{F}_{l_r}, 1)$ where \mathbb{F}_{l_r} is a plane in \mathbb{E}_j . Therefore, according to Proposition 5.2.1, via the previous isomorphism, we have a horizontal curve in $\tilde{\gamma}_{l_r} : [0, T_{l_r}] \rightarrow SO(\mathbb{F}_{l_r}, 1)$ arc-length parameterized whose length is θ_j such that $\tilde{\gamma}_{l_r}(0) = Id_{\mathbb{F}_{l_r}}$ and $\tilde{\gamma}_{l_r}(T_{l_r}) = \theta_j J_{l_r}$. In particular $T_{l_r} = |\theta_j|$. We get a curve $\gamma_{l_r} : [0, \theta_j] \rightarrow SO(\mathbb{E}_j, 1)$ of length θ_j , which joins $Id_{\mathbb{E}_j}$ to some element $\theta_j \hat{J}_{l_r}$ of $SO(\mathbb{E}_j, 1)$

$$(\gamma_{l_r})|_{\mathbb{F}_{l_r}} = \tilde{\gamma}_{l_r}, \quad \text{and } (\gamma_{l_r})|_{[\mathbb{F}_{l_r}]^\perp} = id.$$

It follows that the curve γ_j , obtained by concatenation of the family γ_{l_r} for $l_r = 1, \dots, l_j$, is defined on $[0, n_j \theta_j]$, γ_j is an horizontal curve in $SO(\mathbb{E}_j, 1)$, of length $n_j \theta_j$ which joins $Id_{\mathbb{E}_j}$ to A_j . □

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